Deciding the Value of almost non-Zeno Two-Clock Weighted Timed Games

Isa Vialard

MAX PLANCK INSTITUTE
FOR SOFTWARE SYSTEMS

About me

Looking for a post-doc.

Interested in:

- Timed automata
- Well-quasi orders for verification of infinite-state systems

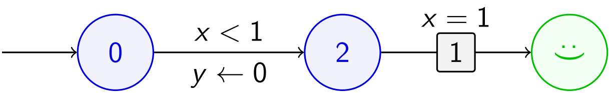
Definition

Weighted Timed Automata

- Clocks $x, y, \ldots$
- Guards $x \bowtie k$ and resets to zero $y \leftarrow 0$
- Weights in $\mathbb{N}$ (or $\mathbb{Z}$) on states & transitions

♣ **Motivations**

Real-time systems, energy consumption



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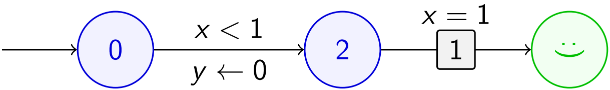
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$



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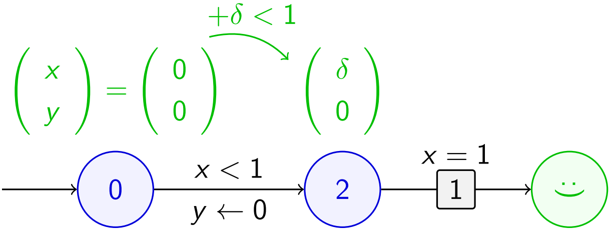

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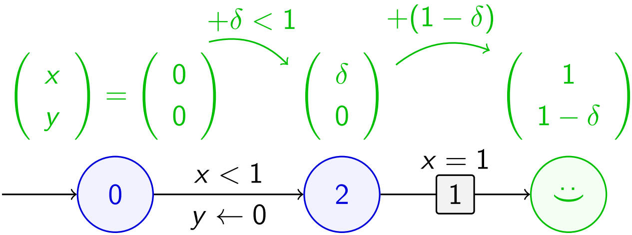


Weight $= (1-\delta) \cdot 2 + 1$

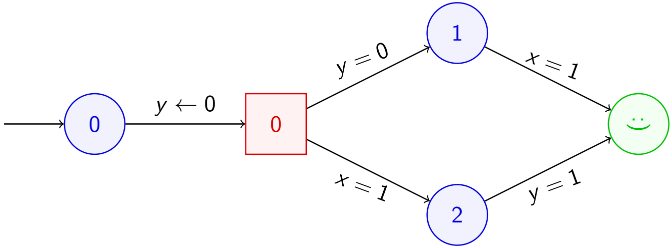
Value $= \inf_{\delta < 1} (1-\delta) \cdot 2 + 1 = 1$

Weighted Timed Games

- States divided between **Min**erva and **Max**imilian
- **Min** wants to reach with minimal cost.

♣ Motivations

Min: Program,
Max: Environment (ext. inputs, user)



Definition

Value problem

$$\mathit{Value}(\mathcal{G}) = \inf_{\sigma_{\mathsf{Min}}} \sup_{\sigma_{\mathsf{Max}}} \mathit{Weight}(\sigma_{\mathsf{Min}}, \sigma_{\mathsf{Max}})$$

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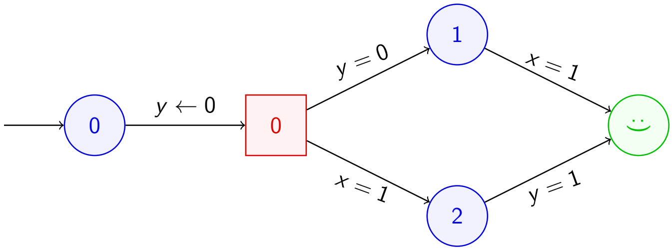

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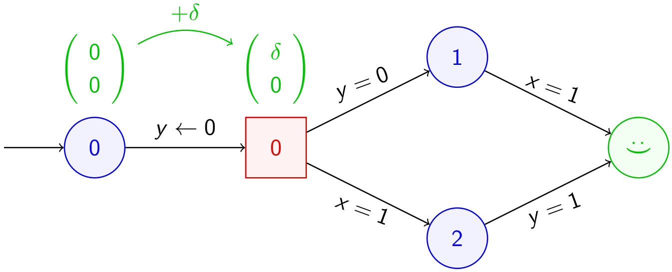

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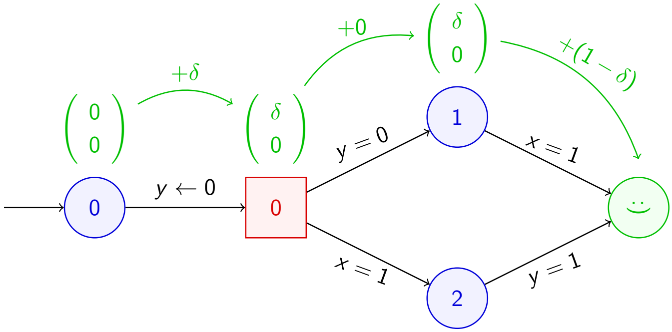

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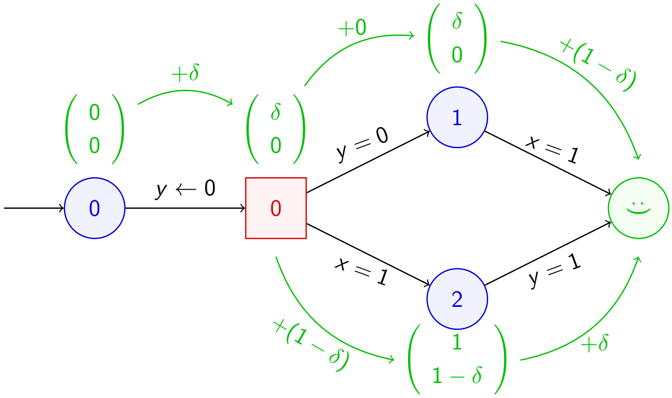

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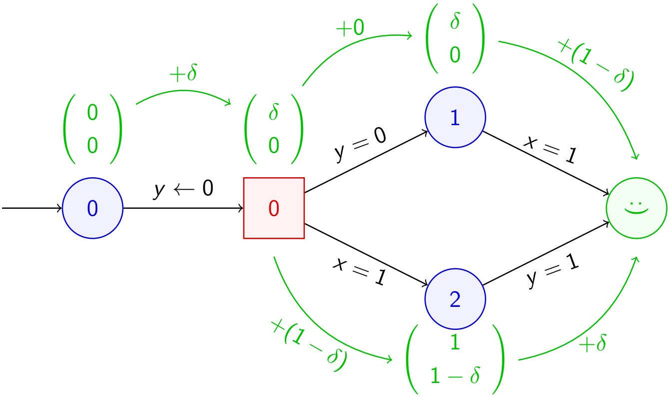
$$\sigma_{\text{Max}} = \max \begin{cases} (1-\delta)\cdot 1 \\ \delta \cdot 2 \end{cases}$$

$\sigma_{\text{Min}} = \inf_\delta \cdots = 1,$

for $\delta = \frac{1}{3}$



State of the art

Number of clocks	Weights in	WTG
1	$\mathbb{N}$	Decidable[2]
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2	$\mathbb{N}$	**Undecidable**[7]
	$\mathbb{Z}$	**Undecidable**
≥ 3	$\mathbb{N}$	Undec.[3]
	$\mathbb{Z}$	Undec.

$\longrightarrow$ Come and see Quentin's talk at FOSSACS

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Number of clocks	Weights in	WTG	*Mysterious subclass?*
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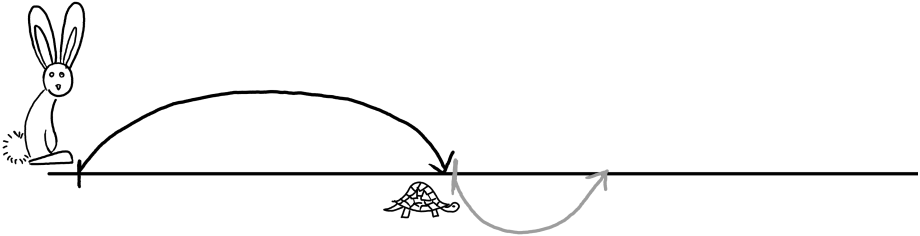
Pathological behaviour: Zeno Paradox



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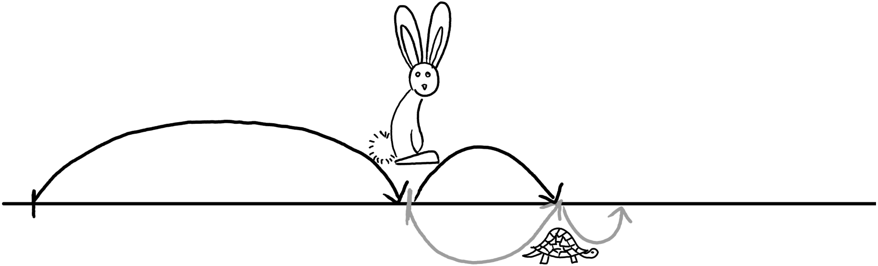

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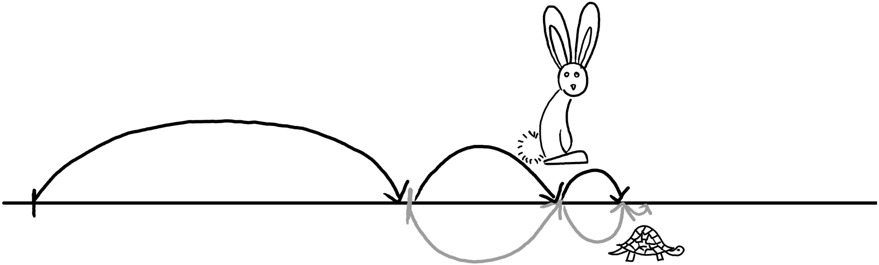
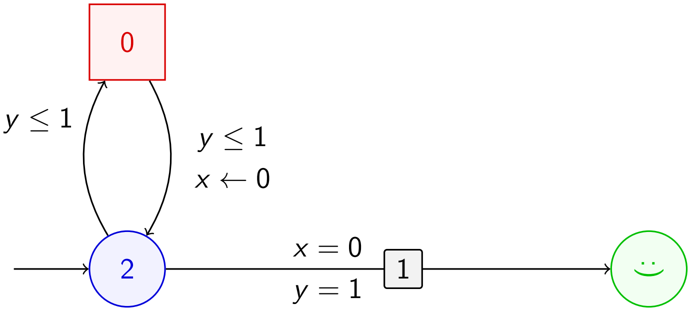


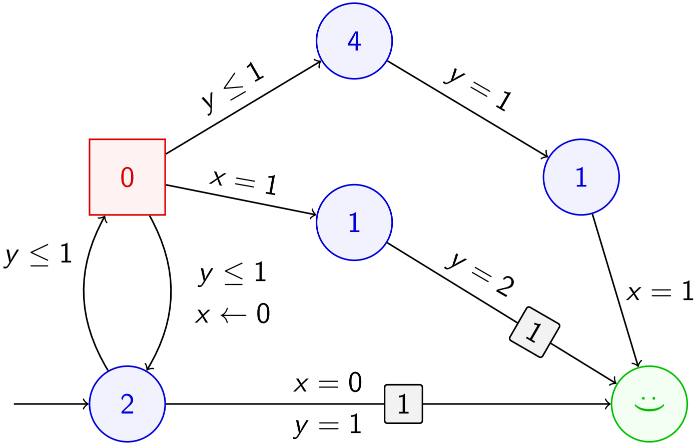
Zeno behaviour: infinitely many transitions in finite time.

A nifty 2-clock gadget



Value = 3

A nifty 2-clock gadget



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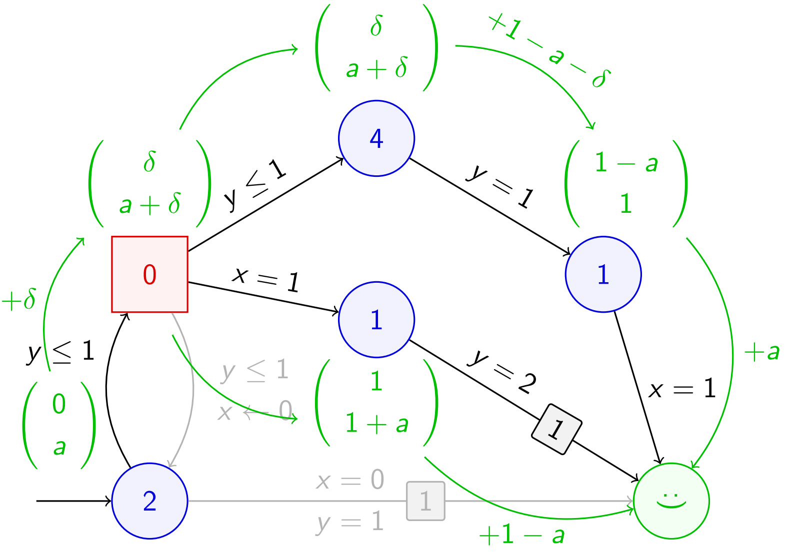
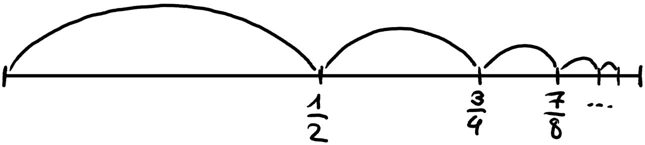


A 2-clock gadget

Min has to pick $\delta = \frac{1-a}{2}$ to minimize $\max \begin{cases} 4(1-a-\delta)+a \\ (1-a)+1 \\ 2(1-a-\delta) \end{cases}$



Our 2-clock gadget: very Zeno

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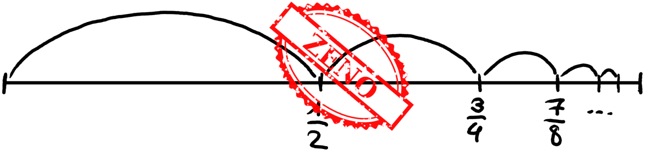

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Definition (Non-Zeno (or divergent) WTG)

Every cycle has weight $\geq 1 \Rightarrow$ finitely many transitions

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Theorem. Non-Zeno $\Rightarrow$ Value decidable

Why? We can unfold cycles into a tree WTG

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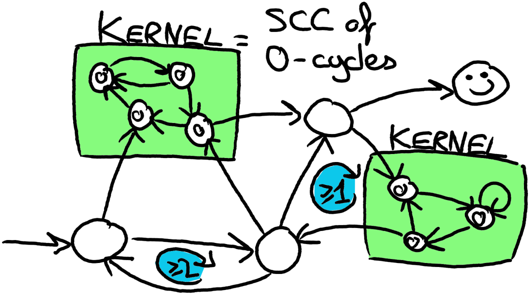
Definition (Non-Zeno (or divergent) WTG)

Every cycle has weight ≥ 1

Definition (Almost non-Zeno WTG)

Every cycle has weight $= 0$ or ≥ 1

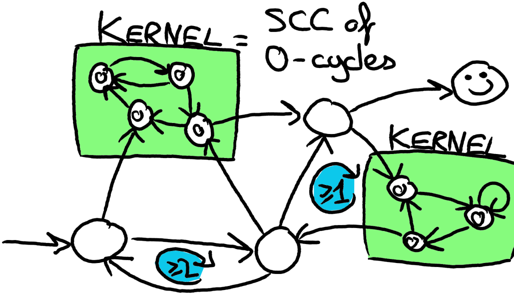


Definition (Non-Zeno (or divergent) WTG)

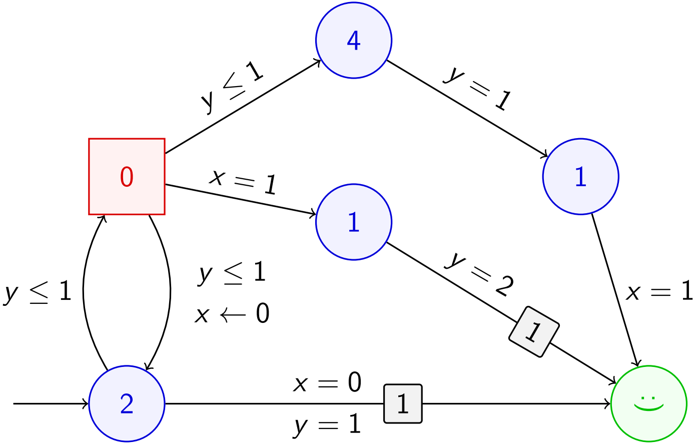
Every cycle has weight ≥ 1

Definition (Almost non-Zeno WTG)

Every cycle has weight $= 0$ or ≥ 1 $\Rightarrow$ Undecidable for ≥ 3 clocks



Our example: not almost non-Zeno



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	$\mathbb{Z}$	**Undec.**	???	Decidable
≥ 3	$\mathbb{N}$	Undec.	**Undec.**[3]	Decidable [1]
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(Almost) Divergence: Partial unfolding

Non-Zeno WTG = decidable, using tree unfolding

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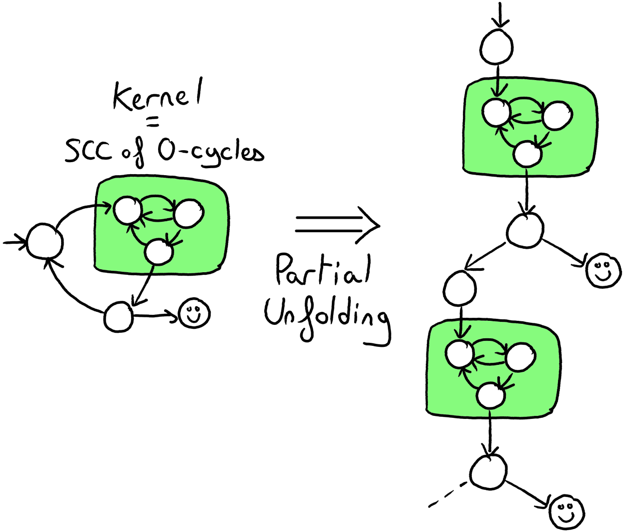
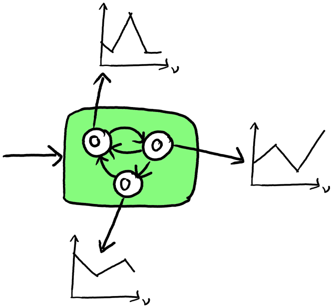
Almost non Zeno → Partial unfolding





What happens in two-clock kernels?

Value iteration algorithm

Build $\texttt{opt}_k$: set of configurations $\rightarrow \mathbb{R}$,

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What happens in two-clock kernels? VIA terminates

VIA terminates: Proof sketch

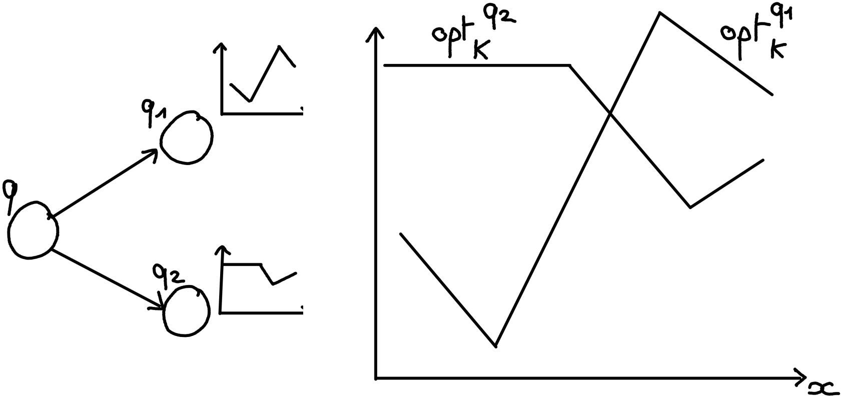
Main problem: $\texttt{opt}_k$ are 2-dimension functions!

$\rightarrow$ Useful simplifications

- Clock values in $[0, 1[$
- Relaxing guards $< \rightarrow \leq$
- **Adding resets everywhere**

VIA terminates: Proof sketch

What form can my opt^k functions take ?



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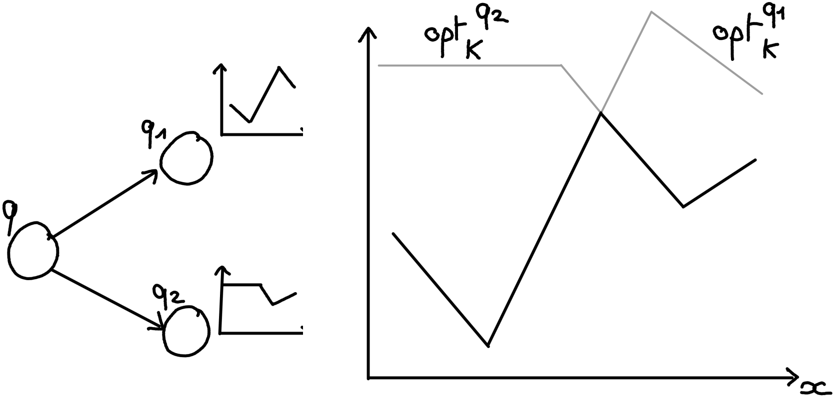

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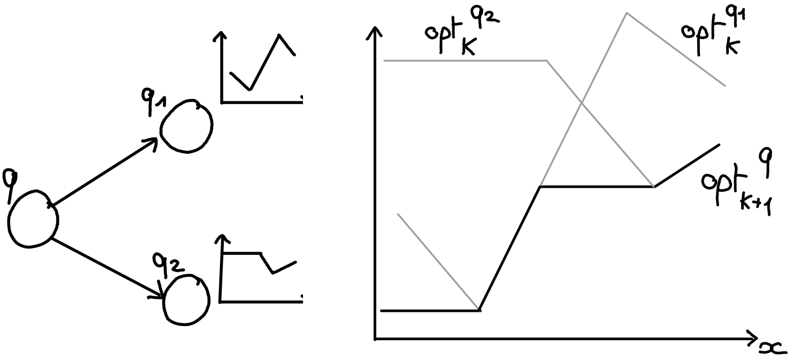

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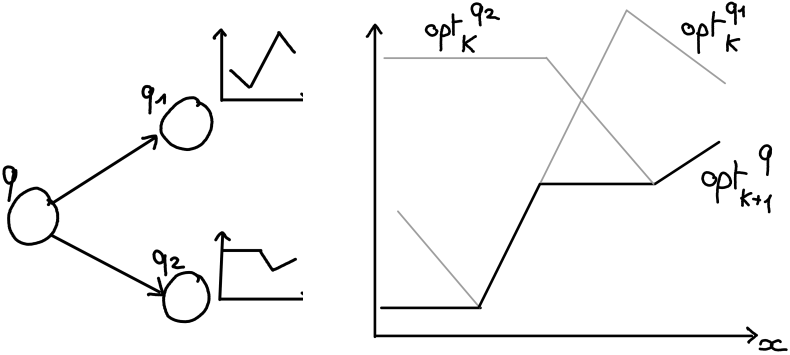

Finite number of combinations: VIA terminates

Conclusion

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Next big question: Approximability

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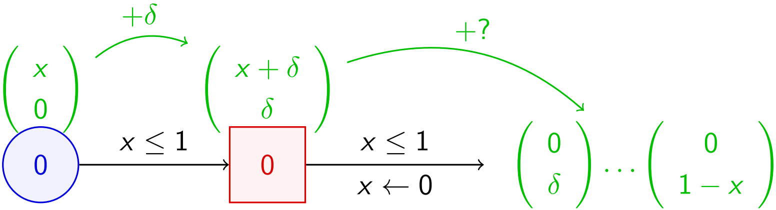
Thank you for listening!

Resets everywere!



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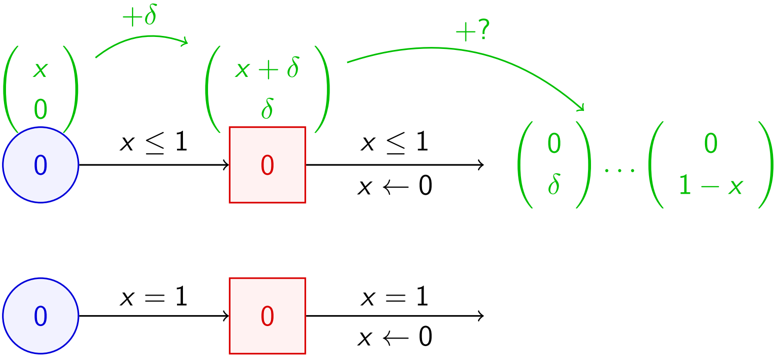

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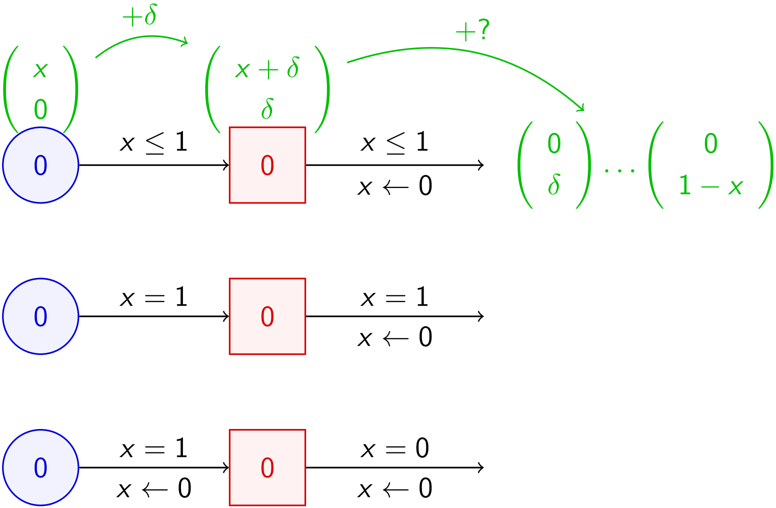

The real “VIA terminates” figure