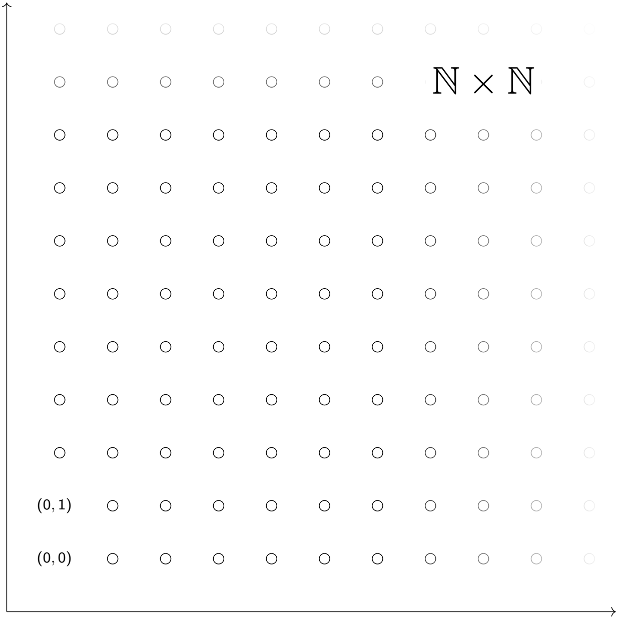
Measuring well quasi-orders and complexity of verification

PhD defense of Isa Vialard

PhD advisor: Philippe Schnoebelen, Directeur de recherche, CNRS, LMF

July 3, 2024

Definitions: Quasi-order



Quasi-order:

reflexive,transitive,
can be partial

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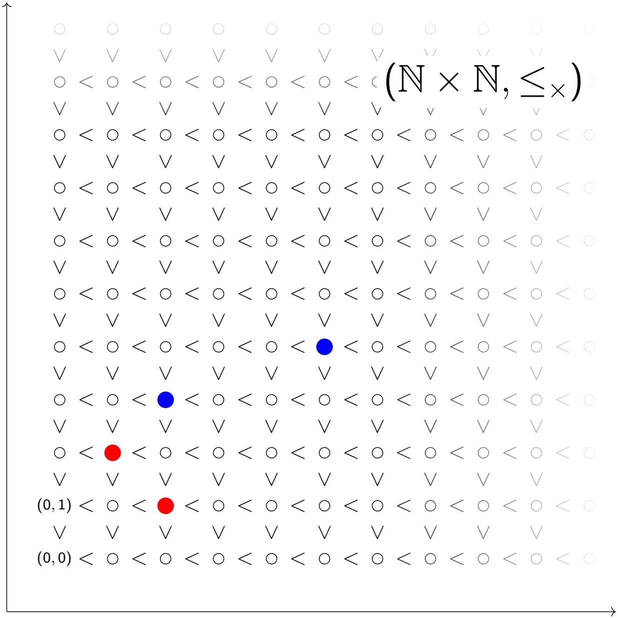


Quasi-order:

reflexive,transitive,

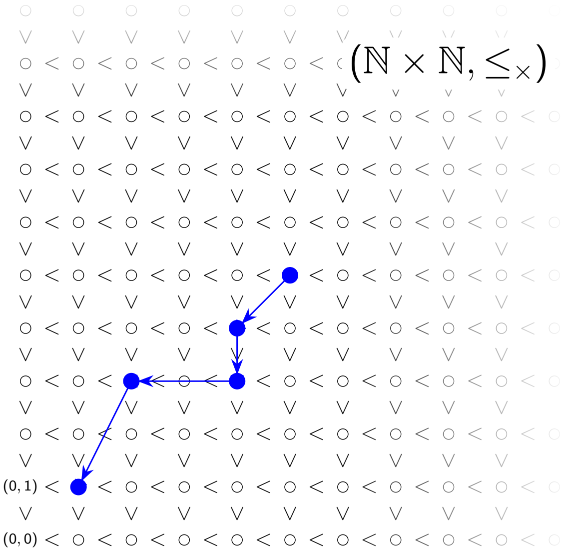
can be partial

Ex: $(2,3) \leq_\times (5,4)$

but $(1,2) \perp (2,1)$

Some interesting sequences

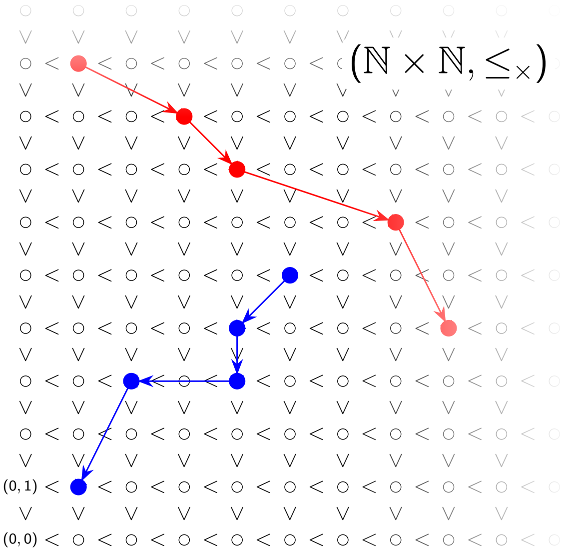
$(\mathbb{N} \times \mathbb{N}, \leq_\times)$



— decreasing sequence

$(5, 5) > (4, 4) > (4, 3)$
$> (2, 3) > (1, 1)$

Some interesting sequences

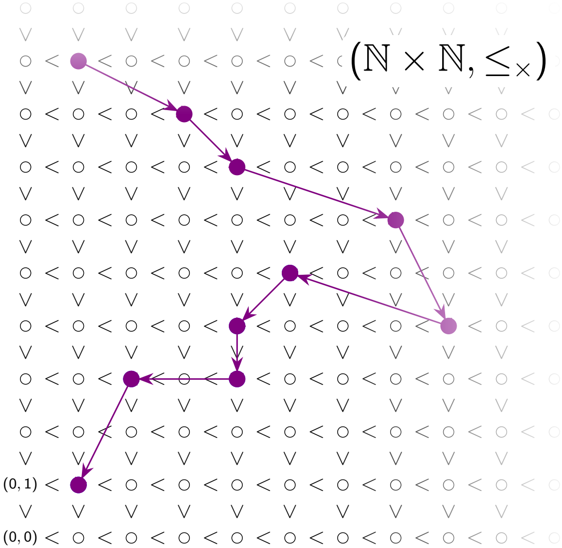


— decreasing sequence

— incomparable sequence
(or antichain)
i.e. pairwise incomparable

$(1,9) \perp (3,8), (4,7), (7,5), \ldots$

Some interesting sequences



— decreasing sequence

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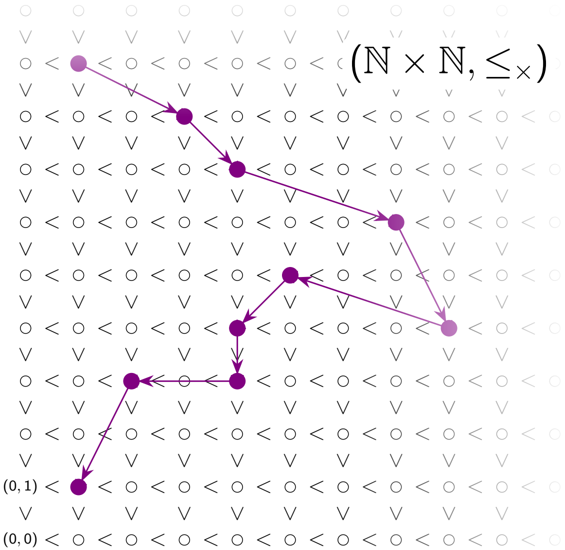
(or antichain)
i.e. pairwise incomparable

— bad sequence
i.e. pairwise non increasing

$(1, 9) \not\leq (3, 8), (4, 7), (7, 5), \ldots$

Definitions: Well Quasi-Order



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WQO

$\Updownarrow$

No infinite antichain
or decreasing seq

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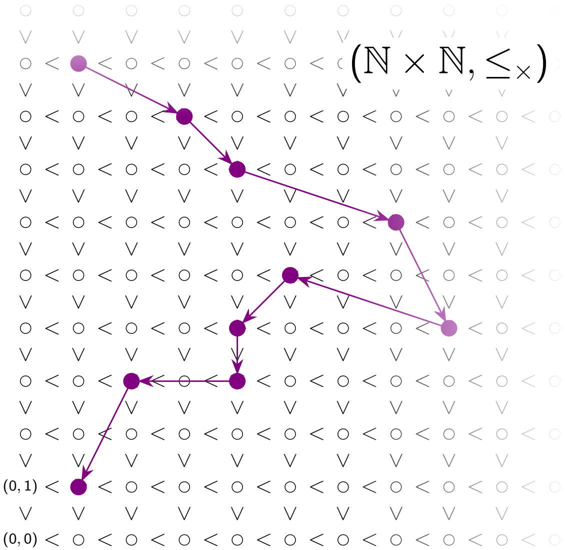

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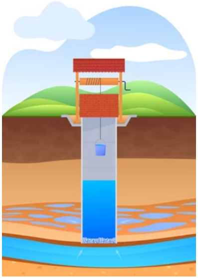

Some see wqos as wells
Blass & Gurevich (2008)

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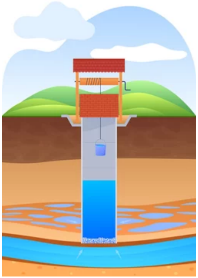

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My parents (2023)

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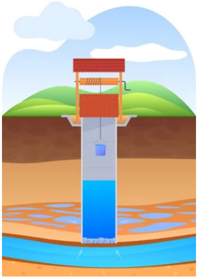

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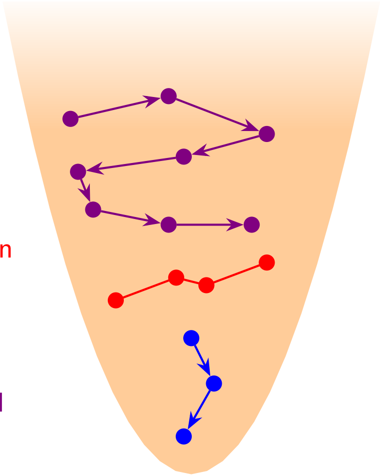

Motivations

♦ Reasons to study wqos

- "It is fun" (Kříž & Thomas (1990))

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♣ Well-structured transition systems

Finkel (1994), Abdulla& Jonsson (1996)

- Set of configurations: WQO
- $\leq$ a simulation relation

$$\begin{array}{ccc} t_1 & \longrightarrow & \color{red}{t_2} \\ \vee\!| & & \color{red}{\vee\!|} \\ s_1 & \longrightarrow & s_2 \end{array}$$

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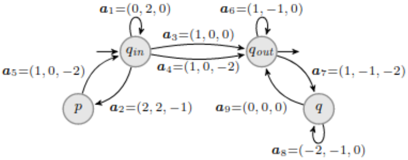
- *Ex: Counter machines, Petri nets, VASS, Lossy channel systems . . .*



Vector Addition Systems with States

♣ Well-structured transition systems

Finkel (1994), Abdulla& Jonsson (1996)

- Set of configurations: WQO
- $\leq$ is *upward-compatible*

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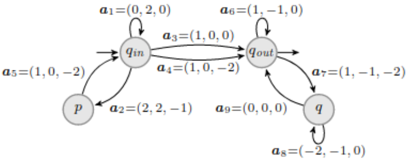

Vector Addition Systems with States

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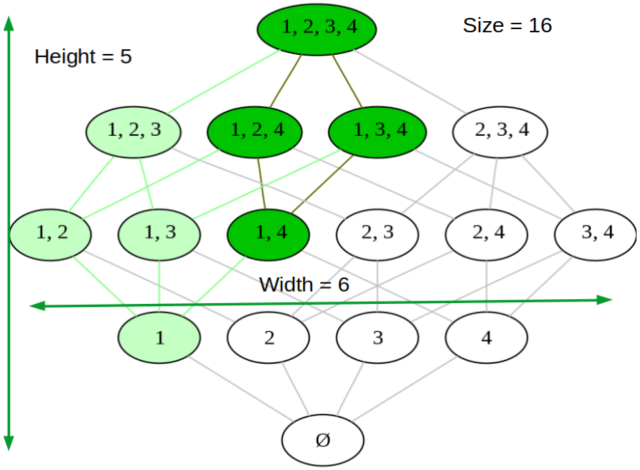
♦ Complexity and expressiveness

Schmitz& Schnoebelen(2011)

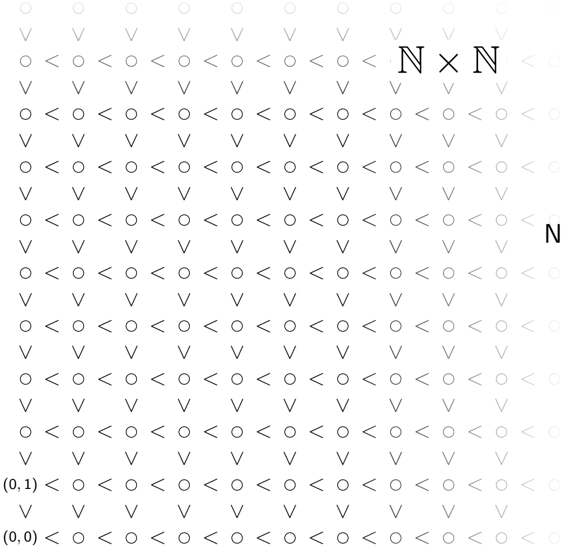
- Controlled bad sequences (even decreasing, or antichains)
- Can we bound the length of controlled sequences by measuring wqo?

♣ Natural notions of measure when finite



Finite subsets of $\{1,2,3,4\}$ ordered by $\subseteq$.

♦ Let's extend height and width to infinite wqos



Problem:

No largest decreasing sequence

No largest antichain

Crash course on ordinal numbers

♦ Enumerate well-orders (i.e. linear wqos)

Cantor(1883)

$$0 \quad < \quad 1 \quad < \quad \ldots \quad < \quad n \quad < \quad \ldots$$

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$$\omega$$

$$\vee$$

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Two operations to build well-orders:

Successor: Add an element on top

Limit: Infinite union of increasing well-orders is well-ordered

$$\omega \quad < \omega + 1 < \quad \dots \quad < \omega + n < \quad \dots$$

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$$\omega \cdot n < \quad \dots$$

$$\vdots$$

$$\omega \cdot 2 < \quad \dots$$

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Ordinal as transitive sets:
$\alpha = \{ \beta < \alpha \}$

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Ordinal as transitive sets:

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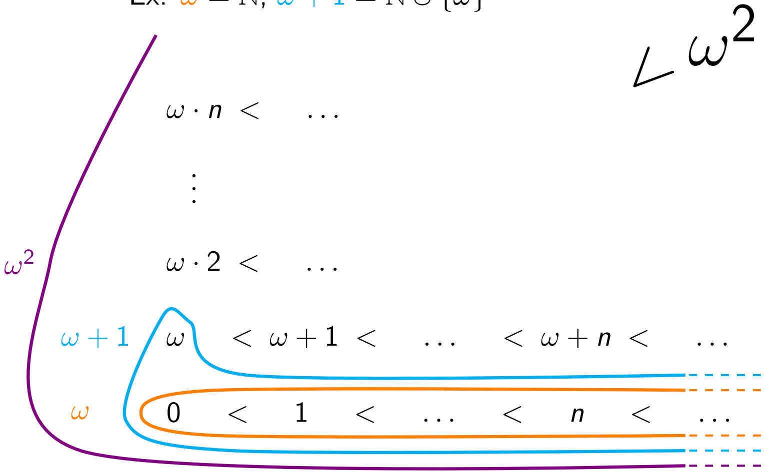
Ex: $\omega = \mathbb{N}$, $\omega + 1 = \mathbb{N} \cup \{\omega\}$



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$$\vdots$$

$$\omega^2 \quad \dots$$

$$\omega \cdot n \; < \quad \dots$$

$$\vdots$$

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$$\omega \; < \; \omega + 1 \; < \; \dots \; < \; \omega + n \; < \; \dots$$

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$$\omega^3$$

$$\vdots \qquad \omega^2 \quad \dots$$

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$\cdots$

ω^n $\cdots$

ω^3

$<$

$\vdots$

ω^2 $\cdots$ $< \omega^\omega$

$<$

$\omega \cdot n < \ \ldots$

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$$< \omega^{\omega} < \omega^{\omega^{\omega}} < \dots < \omega^{\omega^{\cdot^{\cdot^{\omega}}}} < \dots$$

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$$\vdots$$

$$\varepsilon_0$$

$$\vee$$

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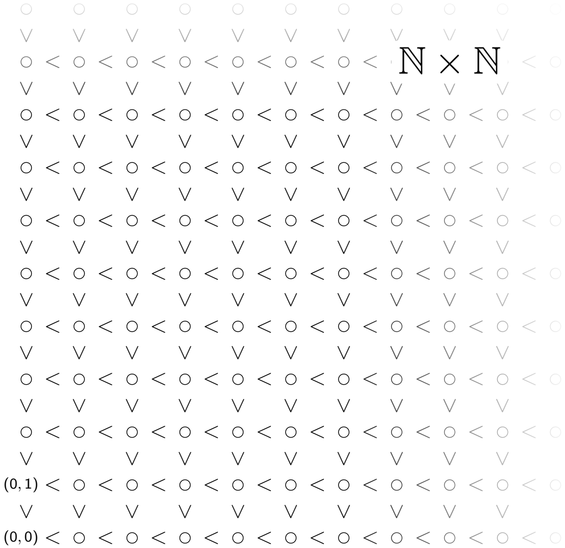
Measuring wqos

♣ **Let's extend height and width to infinite wqos with ordinals**



Width and height:

at least ω

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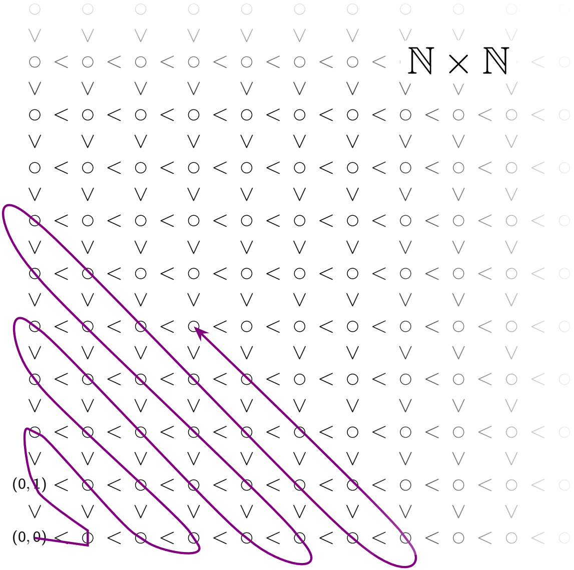

Width and height:

at least ω

Counting elements:

at least ω

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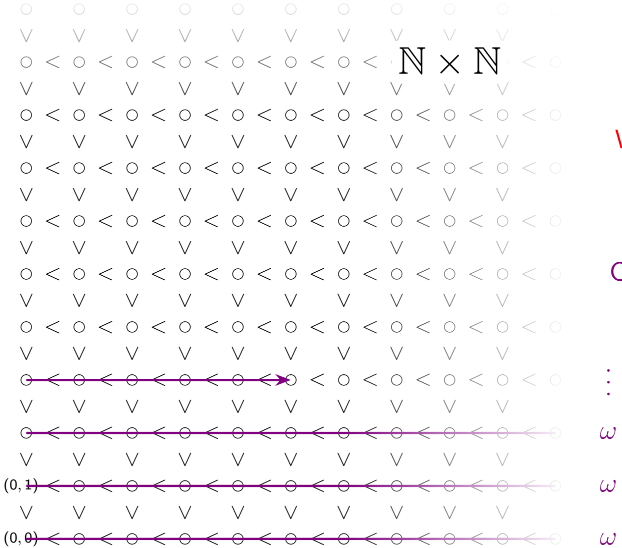


Width and height:

at least ω

Counting elements:

at least ω^2

Definition (Maximal order type, Width and Height)

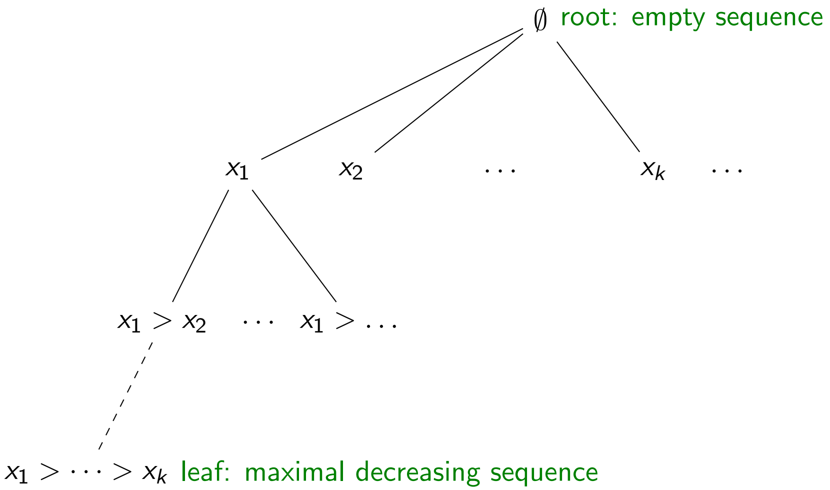
$$\left|\begin{array}{l} \mathrm{o}(X) \\ \mathrm{w}(X) \\ \mathrm{h}(X) \end{array}\right. = \text{ ordinal rank of the tree of } \begin{cases} \text{bad sequences} \\ \text{antichain sequences} \\ \text{decreasing sequences} \end{cases} \text{ in } X.$$

Definition from Kříž & Thomas(1990) (first definition of ordinal width)

First definition of maximal order type by De Jongh & Parikh(1977)

Definition: Rank of well-founded trees

♣ Ex: Tree of decreasing sequences



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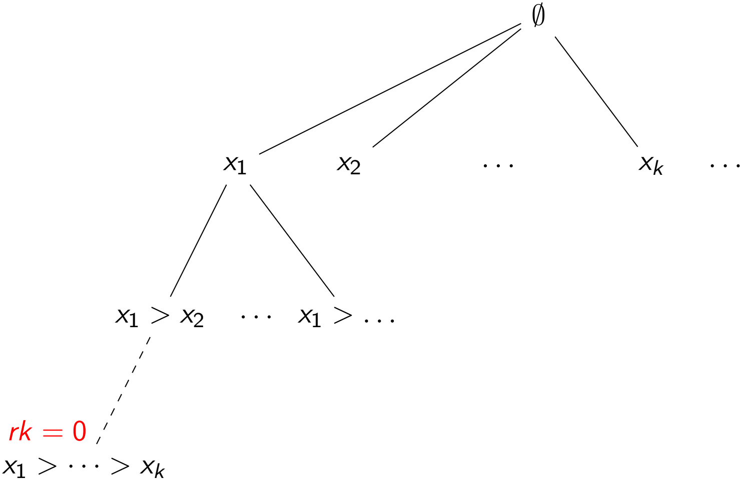

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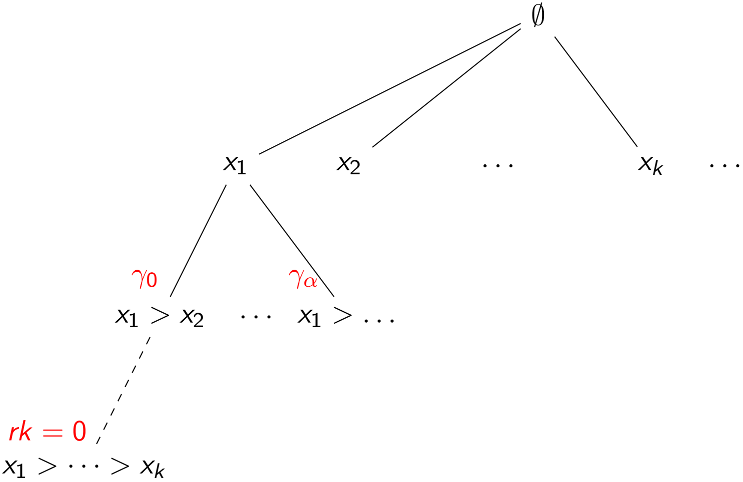

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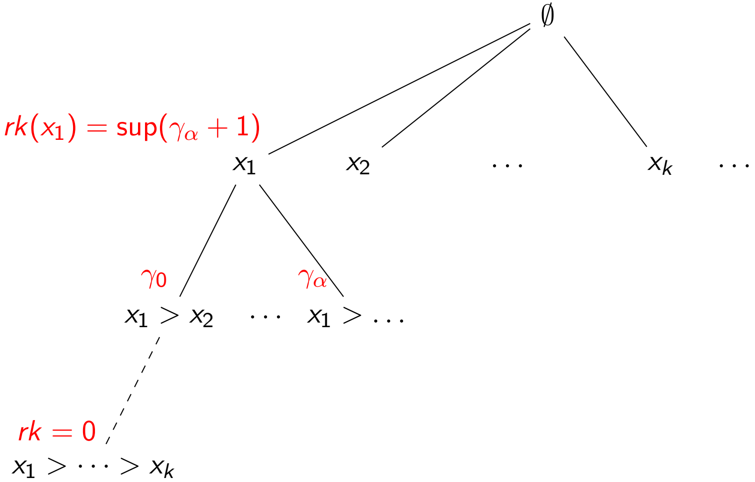

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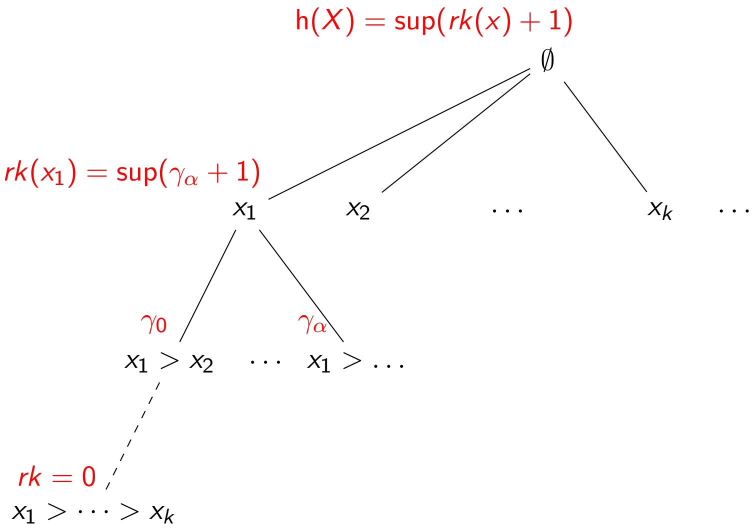

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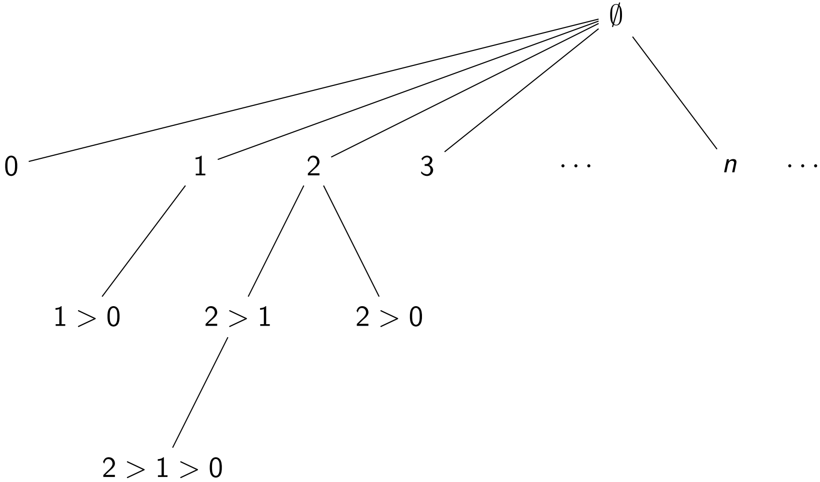
♣ **Ex: Height of $\mathbb{N}$**



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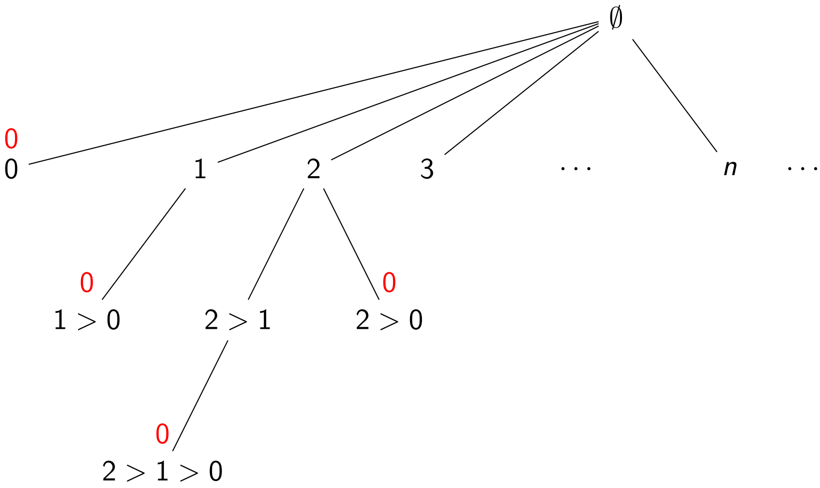

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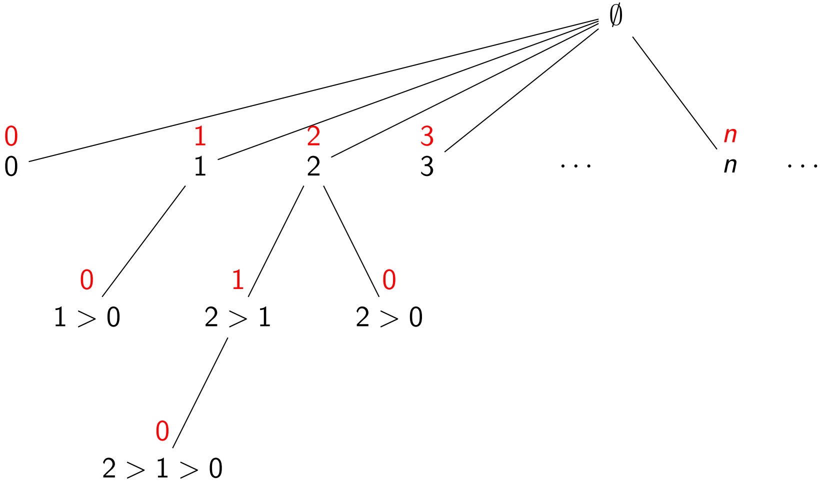

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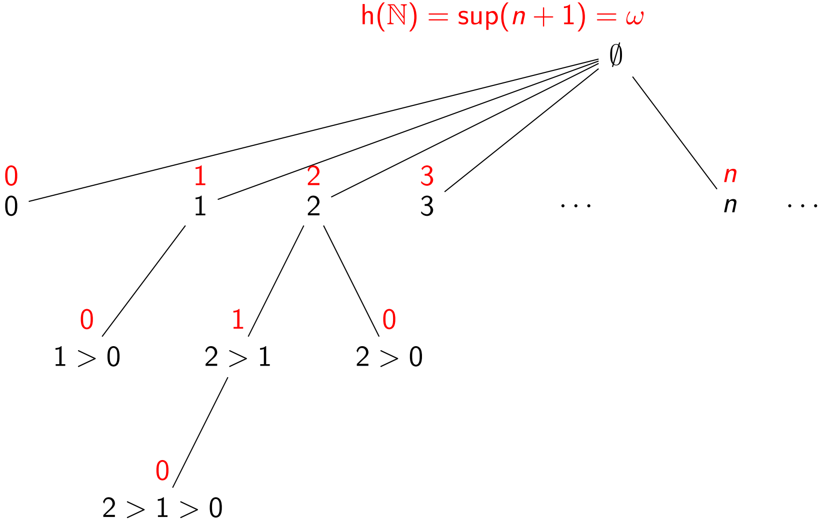
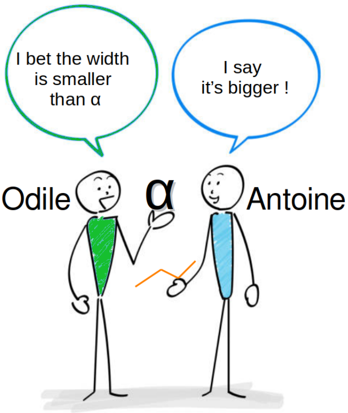
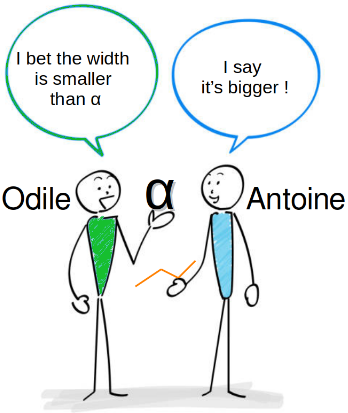


Measuring with games



♦ **Game:** α **vs** w(X)

- Initial configuration:
  - Odile : $\gamma = \alpha$,
  - Antoine : $S = \emptyset$
- Player alternate:
  - Odile picks $\gamma' < \gamma$
  - Antoine extends S into $S :: x$ an antichain,
- End: You lose if you cannot play anymore



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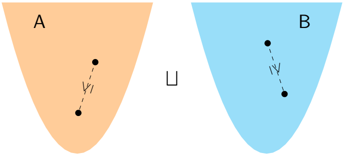
Theorem **(Blass & Gurevich (2008))**

- Antoine has winning strategy when Odile begins $\Leftrightarrow \alpha \leq \mathrm{w}(X)$
- Odile has winning strategy when Antoine begins $\Leftrightarrow \alpha \geq \mathrm{w}(X)$

Example: Playing on the on disjoint sum



Disjoint sum $A \sqcup B$

Theorem: $o(A \sqcup B) = o(A) \oplus o(B)$ **(De Jongh & Parikh(1977))**

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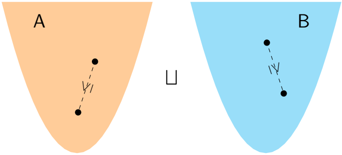

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Ex: $(\omega^\omega + \omega^3) \oplus (\omega^5 + \omega + 1) = \omega^\omega + \omega^5 + \omega^3 + \omega + 1$

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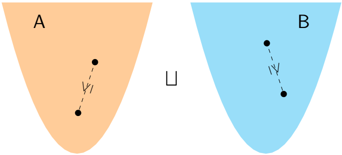

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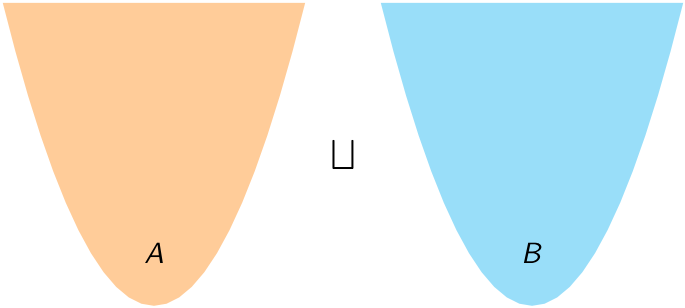
This theorem is easy to prove with games!

Example: Playing on the disjoint sum

Odile Antoine

$o(A) \oplus o(B)$



$o(A \sqcup B) \leq o(A) \oplus o(B)$ if Odile wins when Antoine begins

$o(A \sqcup B) \geq o(A) \oplus o(B)$ if Antoine wins when Odile begins

Example: Playing on the disjoint sum

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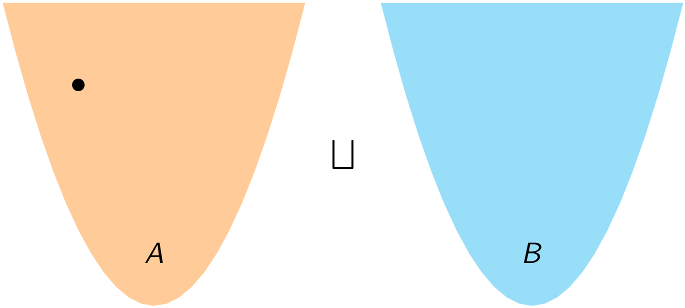
$\mathrm{o}(A) \oplus \mathrm{o}(B)$



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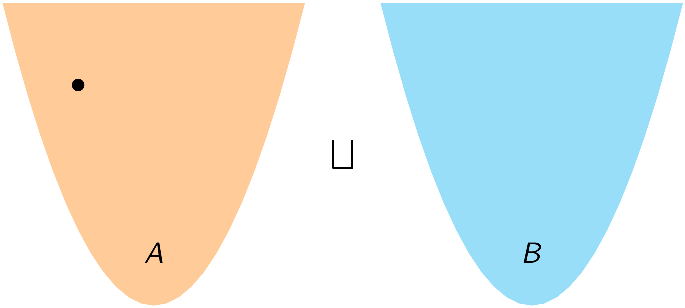
Example: Playing on the disjoint sum

Odile

Antoine

$o(A) \oplus o(B)$

$\alpha_1 \oplus o(B)$



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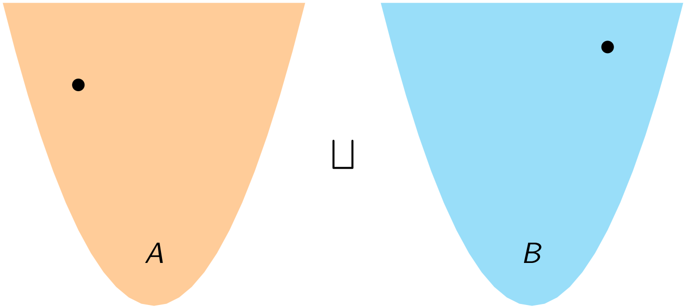


$o(A \sqcup B) \leq o(A) \oplus o(B)$ if Odile wins when Antoine begins

$o(A \sqcup B) \geq o(A) \oplus o(B)$ if Antoine wins when Odile begins

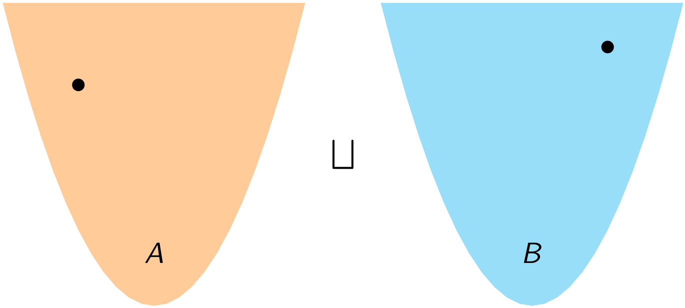
Example: Playing on the disjoint sum

Odile	Antoine

$o(A) \oplus o(B)$

$\alpha_1 \oplus o(B)$

$\alpha_1 \oplus \beta_1$



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Example: Playing on the disjoint sum

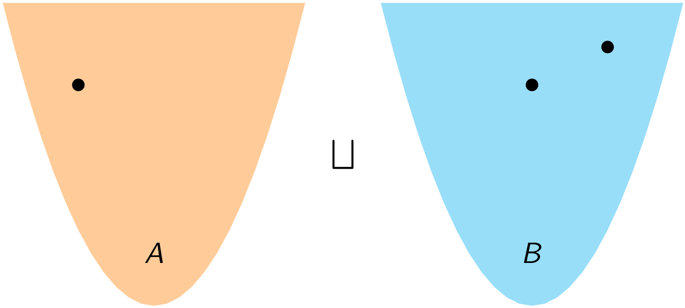
Odile

Antoine

$\mathrm{o}(A) \oplus \mathrm{o}(B)$

$\alpha_1 \oplus \mathrm{o}(B)$

$\alpha_1 \oplus \beta_1$



$\mathrm{o}(A \sqcup B) \leq \mathrm{o}(A) \oplus \mathrm{o}(B)$ if Odile wins when Antoine begins

$\mathrm{o}(A \sqcup B) \geq \mathrm{o}(A) \oplus \mathrm{o}(B)$ if Antoine wins when Odile begins

Example: Playing on the disjoint sum

Odile

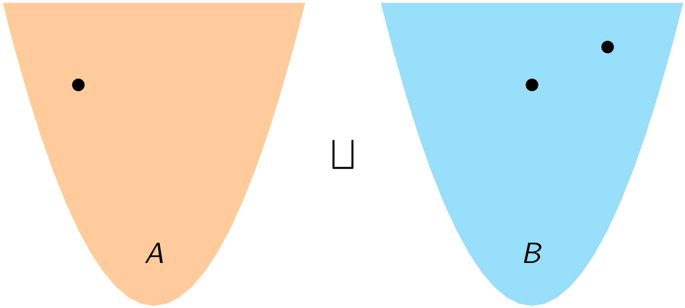
Antoine

$o(A) \oplus o(B)$

$\alpha_1 \oplus o(B)$

$\alpha_1 \oplus \beta_1$

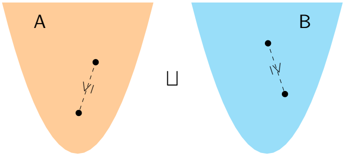
$\alpha_1 \oplus \beta_2$



$o(A \sqcup B) \leq o(A) \oplus o(B)$ if Odile wins when Antoine begins

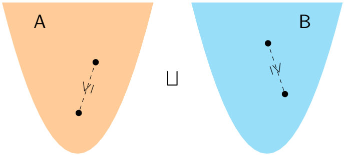
$o(A \sqcup B) \geq o(A) \oplus o(B)$ if Antoine wins when Odile begins

Other classical operations on WQOs

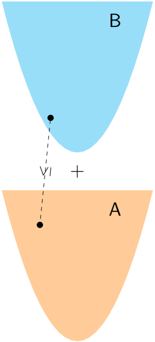


Disjoint sum $A \sqcup B$

Other classical operations on WQOs

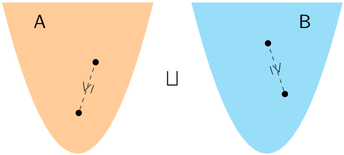


Disjoint sum $A \sqcup B$

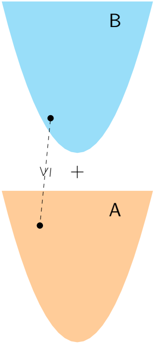


Direct sum $A + B$

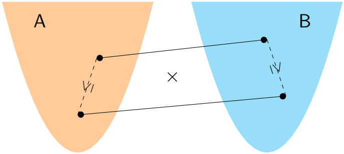
Other classical operations on WQOs



Disjoint sum $A \sqcup B$

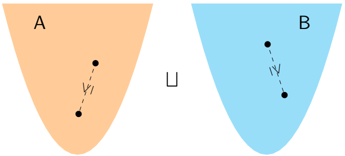


Direct sum $A + B$

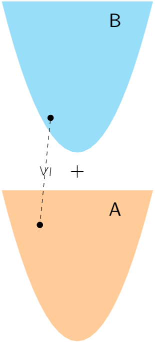


Cartesian product $A \times B$

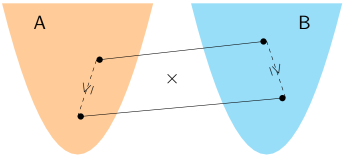
Other classical operations on WQOs



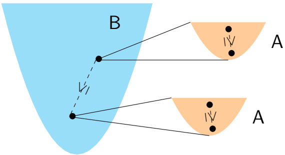
Disjoint sum $A \sqcup B$



Direct sum $A + B$



Cartesian product $A \times B$



Lexicographic product $A \cdot B$

. . . And their ordinal invariants

	Space	M.O.T.	Height	Width
Disjoint sum	$A \sqcup B$	$\mathrm{o}(A) \oplus \mathrm{o}(B)$	$\max(\mathrm{h}(A), \mathrm{h}(B))$	$\mathrm{w}(A) \oplus \mathrm{w}(B)$
Direct sum	$A + B$	$\mathrm{o}(A) + \mathrm{o}(B)$	$\mathrm{h}(A) + \mathrm{h}(B)$	$\max(\mathrm{w}(A), \mathrm{w}(B))$
Cartesian prod.	$A \times B$	$\mathrm{o}(A) \otimes \mathrm{o}(B)$	$\mathrm{h}(A) \hat{\oplus} \mathrm{h}(B)$	**?**
Direct prod.	$A \cdot B$	**?**	$\mathrm{h}(A) \cdot \mathrm{h}(B)$	$\mathrm{w}(A) \odot \mathrm{w}(B)$

... And their ordinal invariants

	Space	M.O.T.	Height	Width
Disjoint sum	$A \sqcup B$	$o(A) \oplus o(B)$	$\max(h(A), h(B))$	$w(A) \oplus w(B)$
Direct sum	$A + B$	$o(A) + o(B)$	$h(A) + h(B)$	$\max(w(A), w(B))$
Cartesian prod.	$A \times B$	$o(A) \otimes o(B)$	$h(A) \hat{\oplus} h(B)$	**?**
Direct prod.	$A \cdot B$	**?**	$h(A) \cdot h(B)$	$w(A) \odot w(B)$
Fin. words	A^*	$\omega^{\omega^{(o(A)^{\pm})}}$	$h^*(A)$	$\omega^{\omega^{(o(A)^{\pm})}}$
Fin. multisets	$M^\diamond(A)$	$\omega^{\widehat{o(A)}}$	$h^*(A)$	**?**
	$M^\circ(A)$	$\omega^{o(A)}$	**?**	**?**
Fin. Powerset	$P_f(A)$	**?**	**?**	**?**

Credits to: De Jongh & Parikh(1977), Schmidt(1979), Abraham & Bonnet(1999), Van der Meeren, Rathjen & Weiermann(2009,2015), Džamonja, Schmitz & Schnoebelen(2020)

My contributions

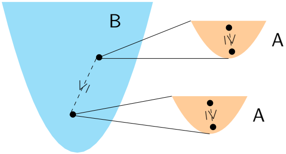
Space	M.O.T.	Height	Width
$A \sqcup B$	$o(A) \oplus o(B)$	$\max(h(A), h(B))$	$w(A) \oplus w(B)$
$A + B$	$o(A) + o(B)$	$h(A) + h(B)$	$\max(w(A), w(B))$
$A \times B$	$o(A) \otimes o(B)$	$h(A) \hat{\oplus} h(B)$	$\geq w(o(A) \times o(B))$
$A \cdot B$	$o(A) \cdot pred_k(o(B)) + o(A) \otimes k$	$h(A) \cdot h(B)$	$w(A) \odot w(B)$
A^*	$\omega^{\omega^{(o(A)^{\pm})}}$	$h^*(A)$	$\omega^{\omega^{(o(A)^{\pm})}}$
$M^\diamond(A)$	$\omega^{\widehat{o(A)}}$	$h^*(A)$	$\omega^{\widehat{o(A)}-1}$
$M^\circ(A)$	$\omega^{o(A)}$	$\omega^{h(A)}$	$\omega^{o_\perp(A)}$
$P_f(A)$	$\leq 2^{o(A)}$	$\leq 2^{h(A)}$	$\geq 2^{w(A)}$

Back in time

Space	M.O.T.	Height	Width
$A \sqcup B$	$\mathrm{o}(A) \oplus \mathrm{o}(B)$	$\max(\mathrm{h}(A), \mathrm{h}(B))$	$\mathrm{w}(A) \oplus \mathrm{w}(B)$
$A + B$	$\mathrm{o}(A) + \mathrm{o}(B)$	$\mathrm{h}(A) + \mathrm{h}(B)$	$\max(\mathrm{w}(A), \mathrm{w}(B))$
$A \times B$	$\mathrm{o}(A) \otimes \mathrm{o}(B)$	$\mathrm{h}(A) \hat{\oplus} \mathrm{h}(B)$	**?**
$A \cdot B$	**?**	$\mathrm{h}(A) \cdot \mathrm{h}(B)$	$\mathrm{w}(A) \odot \mathrm{w}(B)$
A^*	$\omega^{\omega^{(\mathrm{o}(A)^{\pm})}}$	$\mathrm{h}^*(A)$	$\omega^{\omega^{(\mathrm{o}(A)^{\pm})}}$
$\mathrm{M}^{\diamond}(A)$	$\omega^{\widehat{\mathrm{o}(A)}}$	$\mathrm{h}^*(A)$	**?**
$\mathrm{M}^{\circ}(A)$	$\omega^{\mathrm{o}(A)}$	**?**	**?**
$\mathrm{P_f}(A)$	**?**	**?**	**?**

Credits to: De Jongh & Parikh(1977), Schmidt(1979), Abraham & Bonnet(1999), Van der Meeren, Rathjen & Weiermann(2009,2015), Džamonja, Schmitz & Schnoebelen(2020)

Quick look at the direct product



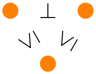
Lexicographic product $A \cdot B$

♦ **I was told that** $o(A \cdot B) = o(A) \cdot o(B)$

…but only the lower bound is true: $o(A \cdot B) \geq o(A) \cdot o(B)$

Mistake noticed by Harry Altman (March, 2024)

Quick look at the direct product

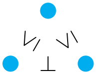


∇

$o = 3$

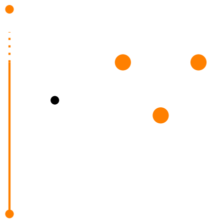
$h = 2$

$w = 2$

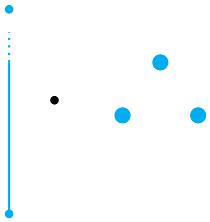


Δ

Quick look at the direct product

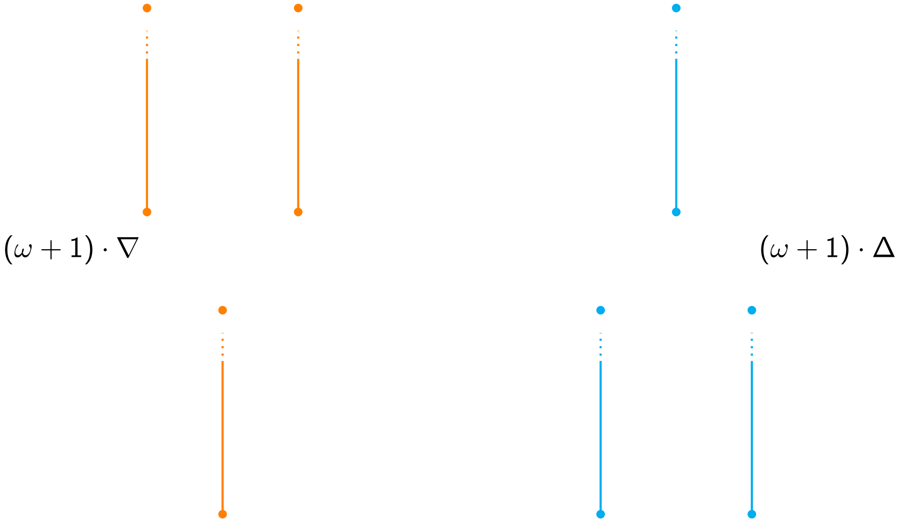


$(\omega + 1) \cdot \nabla$



$(\omega + 1) \cdot \Delta$

Quick look at the direct product



Quick look at the direct product

$$\mathsf{o}((\omega+1)\cdot\nabla) =$$

$$[(\omega+1) \quad \oplus \quad (\omega+1)]$$

$$+$$

$$(\omega+1)$$

$$= \omega\cdot 3+2$$

$$\mathsf{o}((\omega+1)\cdot\Delta) =$$

$$(\omega+1)$$

$$+$$

$$[(\omega+1) \quad \oplus \quad (\omega+1)]$$

$$= \omega\cdot 3+1$$
$$= \mathsf{o}(\omega+1)\cdot\mathsf{o}(\nabla)$$

What about the other operations?

Space	M.O.T.	Height	Width
$A \sqcup B$	$o(A) \oplus o(B)$	$\max(h(A), h(B))$	$w(A) \oplus w(B)$
$A + B$	$o(A) + o(B)$	$h(A) + h(B)$	$\max(w(A), w(B))$
$A \times B$	$o(A) \otimes o(B)$	$h(A) \hat{\oplus} h(B)$	**?**
$A \cdot B$	*Not functional*	$h(A) \cdot h(B)$	$w(A) \odot w(B)$
A^*	$\omega^{\omega^{(o(A)^{\pm})}}$	$h^*(A)$	$\omega^{\omega^{(o(A)^{\pm})}}$
$M^\diamond(A)$	$\omega^{\widehat{o(A)}}$	$h^*(A)$	**?**
$M^\circ(A)$	$\omega^{o(A)}$	**?**	**?**
$P_f(A)$	**?**	**?**	**?**

Credits to: De Jongh & Parikh(1977), Schmidt(1979), Abraham & Bonnet(1999), Van der Meeren, Rathjen & Weiermann(2009,2015), Džamonja, Schmitz & Schnoebelen(2020)

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Space	M.O.T.	Height	Width
$A \sqcup B$	$\mathrm{o}(A) \oplus \mathrm{o}(B)$	$\max(\mathrm{h}(A), \mathrm{h}(B))$	$\mathrm{w}(A) \oplus \mathrm{w}(B)$
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$A \times B$	$\mathrm{o}(A) \otimes \mathrm{o}(B)$	$\mathrm{h}(A) \hat{\oplus} \mathrm{h}(B)$	*Not functional*
$A \cdot B$	*Not functional*	$\mathrm{h}(A) \cdot \mathrm{h}(B)$	$\mathrm{w}(A) \odot \mathrm{w}(B)$
A^*	$\omega^{\omega^{(\mathrm{o}(A)^{\pm})}}$	$\mathrm{h}^*(A)$	$\omega^{\omega^{(\mathrm{o}(A)^{\pm})}}$
$\mathrm{M}^\diamond(A)$	$\omega^{\widehat{\mathrm{o}(A)}}$	$\mathrm{h}^*(A)$	**?**
$\mathrm{M}^\circ(A)$	$\omega^{\mathrm{o}(A)}$	**?**	**?**
$\mathrm{P_f}(A)$	**?**	**?**	**?**

Credits to: De Jongh & Parikh(1977), Schmidt(1979), Abraham & Bonnet(1999), Van der Meeren, Rathjen & Weiermann(2009,2015), Džamonja, Schmitz & Schnoebelen(2020)

What about the other operations?

Space	M.O.T.	Height	Width
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$A + B$	$o(A) + o(B)$	$h(A) + h(B)$	$\max(w(A), w(B))$
$A \times B$	$o(A) \otimes o(B)$	$h(A) \hat{\oplus} h(B)$	*Not functional*
$A \cdot B$	*Not functional*	$h(A) \cdot h(B)$	$w(A) \odot w(B)$
A^*	$\omega^{\omega^{(o(A)^{\pm})}}$	$h^*(A)$	$\omega^{\omega^{(o(A)^{\pm})}}$
$M^\diamond(A)$	$\omega^{\widehat{o(A)}}$	$h^*(A)$	$\omega^{\widehat{o(A)}-1}$
$M^\circ(A)$	$\omega^{o(A)}$	**?**	**?**
$P_f(A)$	**?**	**?**	**?**

Credits to: De Jongh & Parikh(1977), Schmidt(1979), Abraham & Bonnet(1999), Van der Meeren, Rathjen & Weiermann(2009,2015), Džamonja, Schmitz & Schnoebelen(2020)

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$A \times B$	$o(A) \otimes o(B)$	$h(A) \hat{\oplus} h(B)$	*Not functional*
$A \cdot B$	*Not functional*	$h(A) \cdot h(B)$	$w(A) \odot w(B)$
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$M^\diamond(A)$	$\omega^{\widehat{o(A)}}$	$h^*(A)$	$\omega^{\widehat{o(A)}-1}$
$M^\circ(A)$	$\omega^{o(A)}$	$\omega^{h(A)}$	**?**
$P_f(A)$	**?**	**?**	**?**

Credits to: De Jongh & Parikh(1977), Schmidt(1979), Abraham & Bonnet(1999), Van der Meeren, Rathjen & Weiermann(2009,2015), Džamonja, Schmitz & Schnoebelen(2020)

What about the other operations?

Space	M.O.T.	Height	Width
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Credits to: De Jongh & Parikh(1977), Schmidt(1979), Abraham & Bonnet(1999), Van der Meeren, Rathjen & Weiermann(2009,2015), Džamonja, Schmitz & Schnoebelen(2020)

What about the other operations?

Space	M.O.T.	Height	Width
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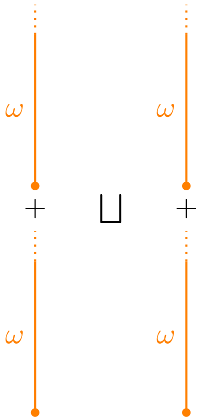
$A \times B$	$o(A) \otimes o(B)$	$h(A) \hat{\oplus} h(B)$	*Not functional*
$A \cdot B$	*Not functional*	$h(A) \cdot h(B)$	$w(A) \odot w(B)$
A^*	$\omega^{\omega^{(o(A)^{\pm})}}$	$h^*(A)$	$\omega^{\omega^{(o(A)^{\pm})}}$
$M^\diamond(A)$	$\omega^{\widehat{o(A)}}$	$h^*(A)$	$\omega^{\widehat{o(A)}-1}$
$M^\circ(A)$	$\omega^{o(A)}$	$\omega^{h(A)}$	*Not functional*
$P_f(A)$	*Not functional*	*Not functional*	*Not functional*

Credits to: De Jongh & Parikh(1977), Schmidt(1979), Abraham & Bonnet(1999), Van der Meeren, Rathjen & Weiermann(2009,2015), Džamonja, Schmitz & Schnoebelen(2020)

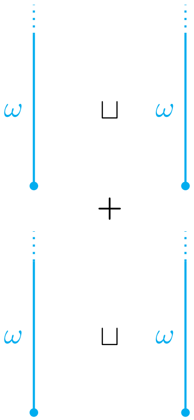
Non functional example for $\mathrm{P_f}$



$$\mathrm{o} = \omega \cdot 4$$

$$\mathrm{h} = \omega \cdot 2$$

$$\mathrm{w} = 2$$

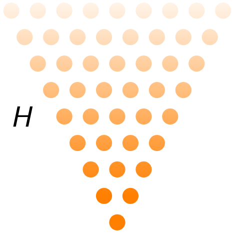


$Y_1 = (\omega + \omega) \sqcup (\omega + \omega)$

$Y_2 = (\omega \sqcup \omega) + (\omega \sqcup \omega)$

$$f(\mathrm{P_f}(Y_1)) \neq f(\mathrm{P_f}(Y_2)) \text{ for } f = \mathrm{o}, \mathrm{h}, \mathrm{w}$$

Non functional example for Cartesian product and multiset ordering

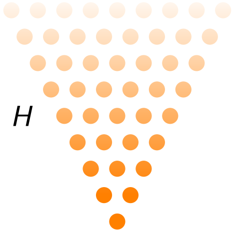


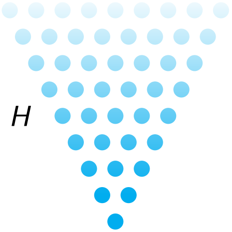
$$\mathsf{o} = \omega \cdot 2$$

$$\mathsf{h} = \omega \cdot 2$$

$$\mathsf{w} = \omega$$

$+$ $+$





$X_1 = H + H$

$X_2 = H + \omega$

$\mathsf{w}(X_1 \times \omega) \neq \mathsf{w}(X_2 \times \omega)$

$\mathsf{w}(\mathsf{M}^{\circ}(X_1)) \neq \mathsf{w}(\mathsf{M}^{\circ}(X_2))$

Non functionality

What can we do?

Dealing with non-functionality

- **Three main approaches**
  - Finding functional (tight) bounds

Dealing with non-functionality

♦ Three main approaches

- Finding functional (tight) bounds

♣ Bounds on the finite powerset

From a joint article with Abriola, Halfon, Lopez, Schmitz, Schnoebelen

$$1 + \mathsf{o}(A) \leq \mathsf{o}(\mathsf{P_f}(A)) \leq 2^{\mathsf{o}(A)}$$
$$1 + \mathsf{h}(A) \leq \mathsf{h}(\mathsf{P_f}(A)) \leq 2^{\mathsf{h}(A)}$$
$$2^{\mathsf{w}(A)} \leq \mathsf{w}(\mathsf{P_f}(A))$$

Dealing with non-functionality

♦ Three main approaches

- Finding functional (tight) bounds

♣ Bounds on the finite powerset

From a joint article with Abriola, Halfon, Lopez, Schmitz, Schnoebelen

$$1 + \mathrm{o}(A) \leq \mathrm{o}(\mathrm{P_f}(A)) \leq 2^{\mathrm{o}(A)}$$
$$1 + \mathrm{h}(A) \leq \mathrm{h}(\mathrm{P_f}(A)) \leq 2^{\mathrm{h}(A)}$$
$$2^{\mathrm{w}(A)} \leq \mathrm{w}(\mathrm{P_f}(A))$$

Hence $2^{\mathrm{w}(A)} \leq \mathrm{w}(\mathrm{P_f}(A)) \leq \mathrm{o}(\mathrm{P_f}(A)) \leq 2^{\mathrm{o}(A)}$

Dealing with non-functionality

♦ Three main approaches

- Finding functional (tight) bounds

♣ Bounds on the finite powerset

From a joint article with Abriola, Halfon, Lopez, Schmitz, Schnoebelen

$$1 + \mathrm{o}(A) \leq \mathrm{o}(\mathrm{P_f}(A)) \leq 2^{\mathrm{o}(A)}$$
$$1 + \mathrm{h}(A) \leq \mathrm{h}(\mathrm{P_f}(A)) \leq 2^{\mathrm{h}(A)}$$
$$2^{\mathrm{w}(A)} \leq \mathrm{w}(\mathrm{P_f}(A))$$

Hence $2^{\mathrm{w}(A)} = \mathrm{w}(\mathrm{P_f}(A)) = \mathrm{o}(\mathrm{P_f}(A)) = 2^{\mathrm{o}(A)}$ when $\mathrm{w}(A) = \mathrm{o}(A)$

Dealing with non-functionality

♦ Three main approaches

- Finding functional (tight) bounds
- Delimiting a wide family of well-behaved wqos

  Ex: Wqos that verify $w = o$, Cartesian product of ordinals

Dealing with non-functionality

♦ Three main approaches

- Finding functional (tight) bounds
- Delimiting a wide family of well-behaved wqos

  Ex: Wqos that verify $\mathrm{w} = \mathrm{o}$, Cartesian product of ordinals
- *the third one will amaze you!*

Bounding ordinal invariants

Upper bounds

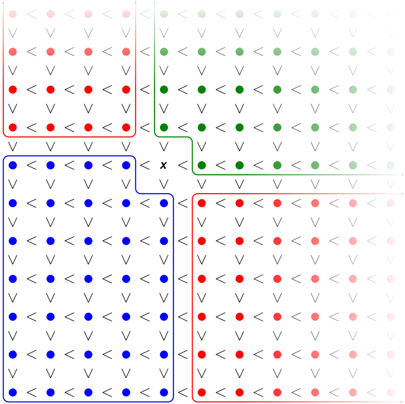
Finding upper bound : Residual equations

♦ **Residuals of a wqo**

$$A_{<x} = \{\, y \in A \mid y < x \,\}$$
$$A_{\perp x} = \{\, y \in A \mid y \perp x \,\}$$
$$A_{>x} = \{\, y \in A \mid y \not\geq x \,\}$$
$$A_{\not\geq x} = \{\, y \in A \mid y \not\geq x \,\}$$
$$= A_{<x} \cup A_{\perp x}$$

♣ **Ex: Residuals of $\mathbb{N} \times \mathbb{N}$**

. **a.k.a.** $\omega \times \omega$



Finding upper bound : Residual equations

♦ **Residual equations**

$$\mathrm{o}(A) = \sup_{x \in A} \mathrm{o}(A_{\not\geq x}) + 1$$

$$\mathrm{h}(A) = \sup_{x \in A} \mathrm{h}(A_{<x}) + 1$$

$$\mathrm{w}(A) = \sup_{x \in A} \mathrm{w}(A_{\perp x}) + 1$$

♣ **Ex: Residuals of $\mathbb{N} \times \mathbb{N}$**

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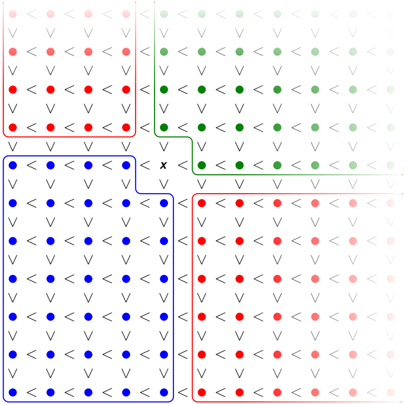
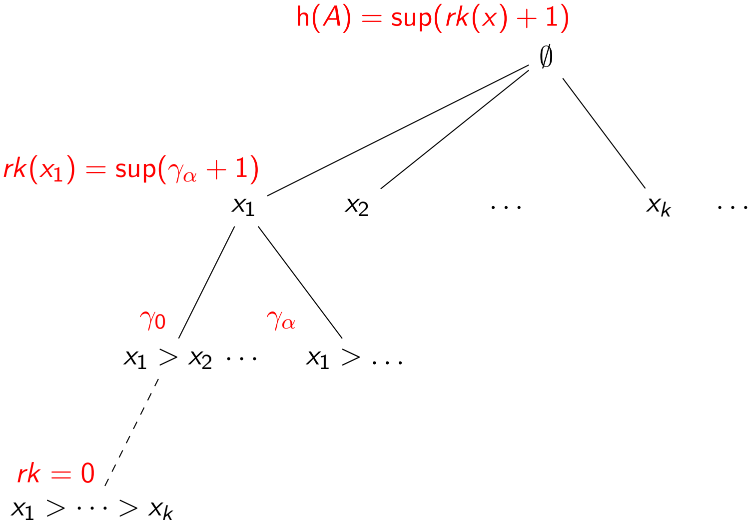


Link with tree rank definition



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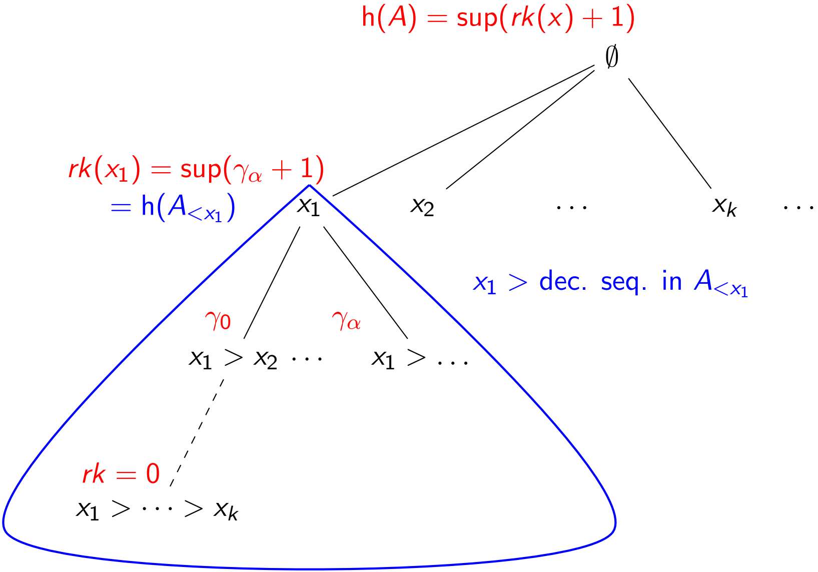
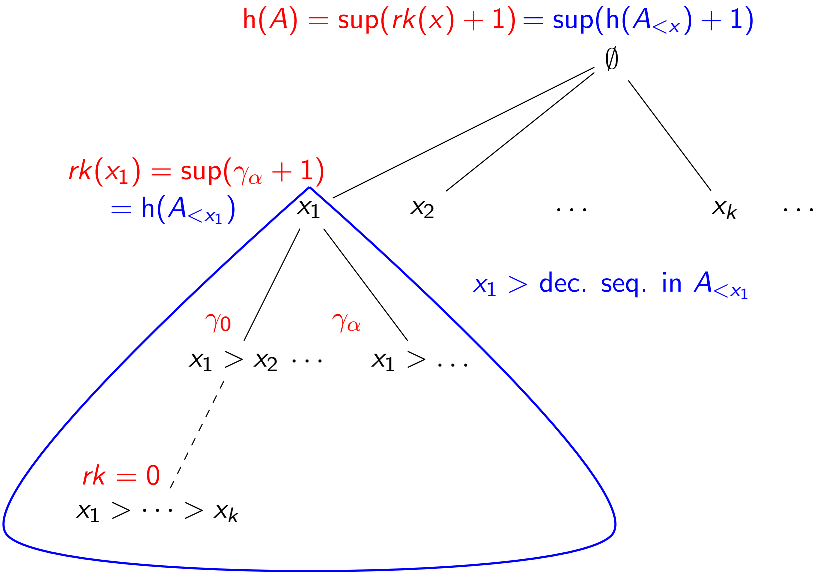


Link with tree rank definition

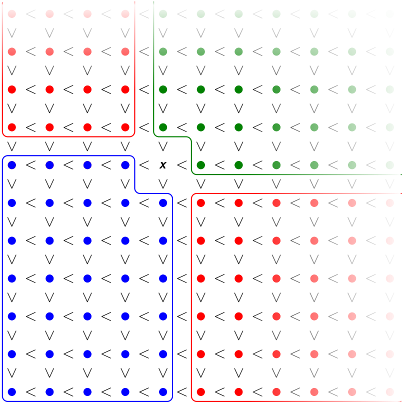


♦ Residual equations

$$o(A) = \sup_{x \in A} o(A_{\not\geq x}) + 1$$

$$h(A) = \sup_{x \in A} h(A_{<x}) + 1$$

$$w(A) = \sup_{x \in A} w(A_{\perp x}) + 1$$

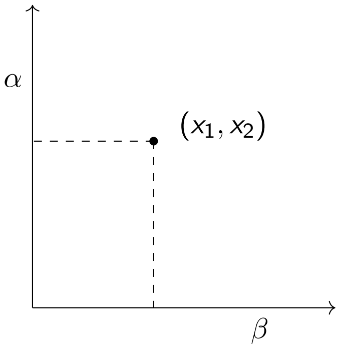


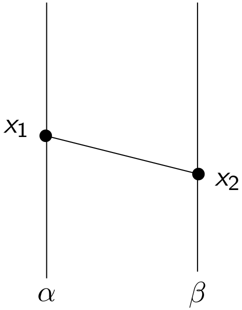
♣ Properties

- $o(A_{\not\geq x}) < o(A)$,
- $h(A_{<x}) < h(A)$, $o(A_{<x}) < o(A)$
- $w(A_{\perp x}) < w(A)$, $o(A_{\perp x}) < o(A)$
- However, $(\mathbb{N} \times \mathbb{N})_{>x}$ contains a copy of $\mathbb{N} \times \mathbb{N}$

Example: Using the residual equations

♣ **How to compute $w(\alpha \times \beta)$** **(From Abraham (1987))**





$\alpha \times \beta$

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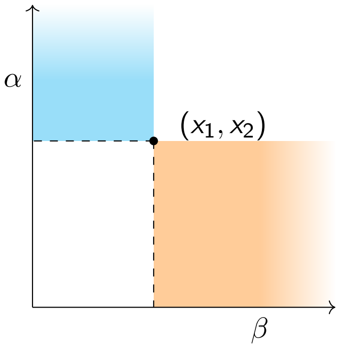

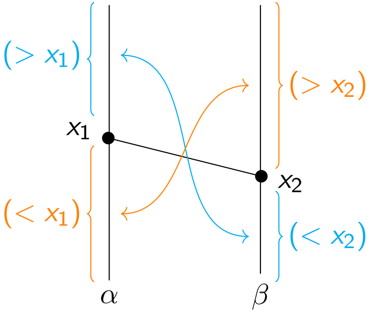

$\alpha \times \beta$

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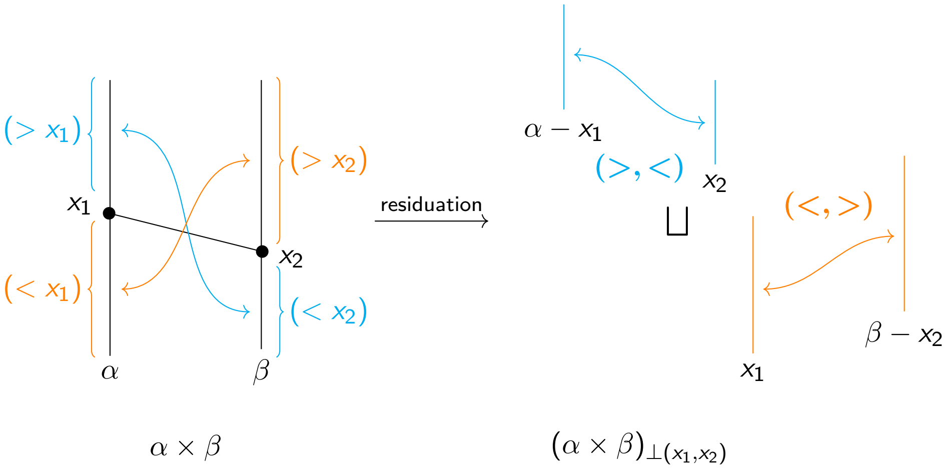
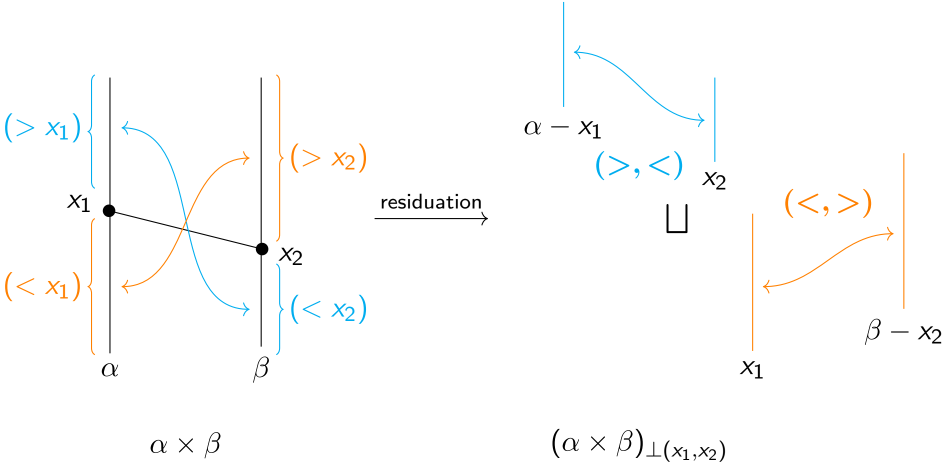


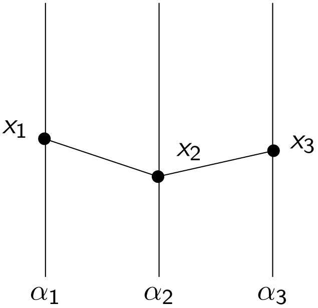
Example: Using the residual equations

♣ **How to compute $w(\alpha \times \beta)$** (From Abraham (1987))

$$w(\alpha \times \beta) = \sup_{x_1, x_2}(w((\alpha - x_1) \times x_2) \oplus w(x_1 \times (\beta - x_2)) + 1)$$



Same method, for three ordinals



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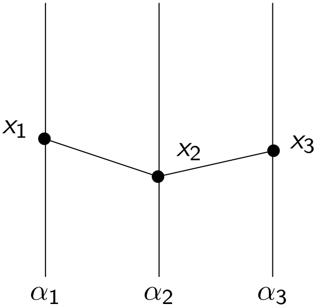

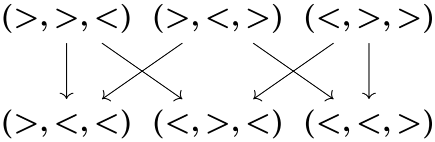

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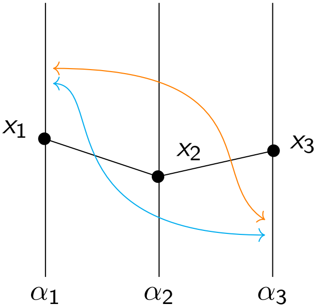
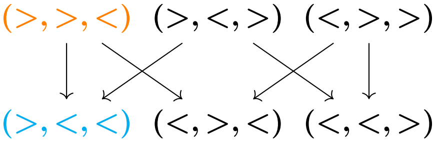
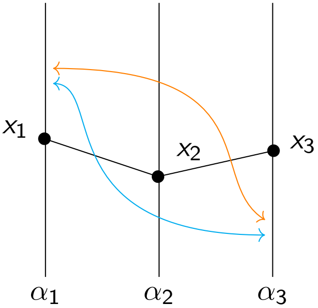


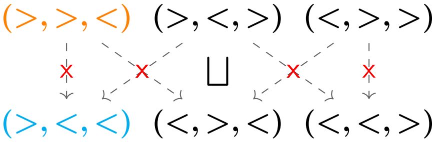


Same method, for three ordinals

$$\mathrm{w}(\alpha_1 \times \alpha_2 \times \alpha_3) \leq \sup_{x_1,x_2,x_3}(\mathrm{w}((\alpha_1 - x_1) \times (\alpha_2 - x_2) \times x_3)$$
$$\oplus(\mathrm{w}((\alpha_1 - x_1) \times x_2 \times x_3) \oplus \cdots + 1)$$



$\leq_{\boldsymbol{w}}$



$$w(\alpha_1 \times \alpha_2 \times \alpha_3) \leq \sup_{x_1,x_2,x_3}(w((\alpha_1 - x_1) \times (\alpha_2 - x_2) \times x_3)$$
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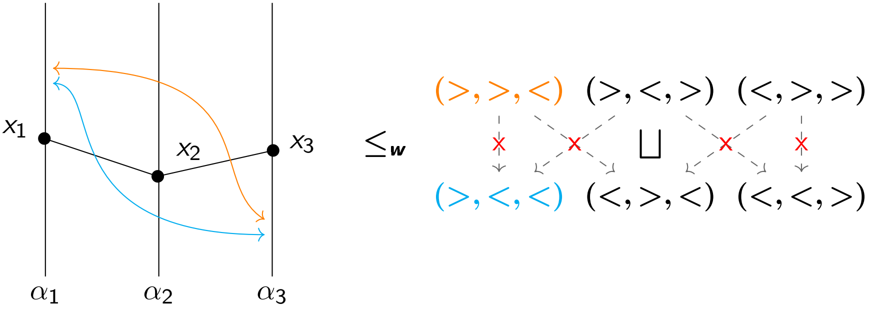
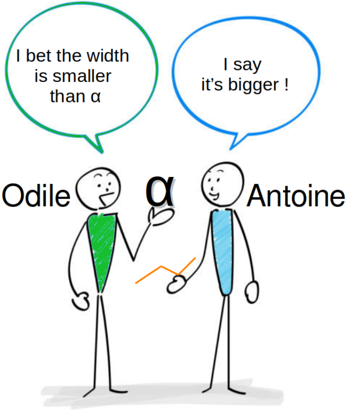


♦ **The method of residuals provides an upper bound. . .**

How can we prove a lower bound ?

Bounding ordinal invariants

Lower bounds



♦ **Game:** α **vs** w(X)

- Initial configuration:
  - Odile : $\gamma = \alpha$,
  - Antoine : $S = \emptyset$
- Each turn:
  - Odile : $\gamma \leftarrow \gamma' < \gamma$
  - Antoine : $S \leftarrow S :: x$,

  Requires: S antichain
- End: First one who can't play loses!

♣ **Lower bound: we want a winning strategy for Antoine**

Reasoning with games: Slices

- Imagine this is a wqo...

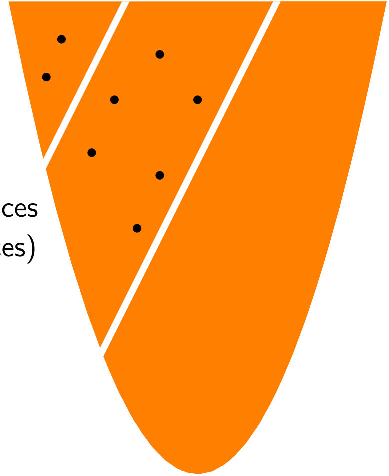
Reasoning with games: Slices

Slice X into disjoint subsets whose width is known (Antoine has a winning strategy)

Reasoning with games: Slices

Slice X into disjoint subsets whose width is known (Antoine has a winning strategy)

Can Antoine combine his strategies on the slices into a winning strategy on X against $\sum$ w(slices)



Reasoning with games: Slices

Slice X into disjoint subsets whose width
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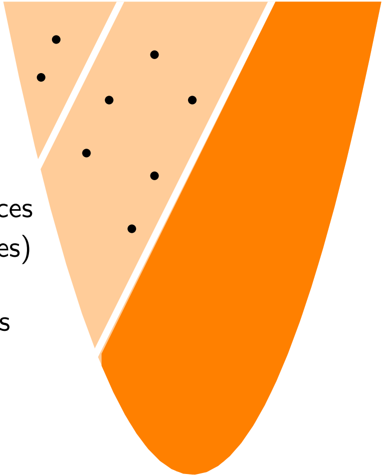
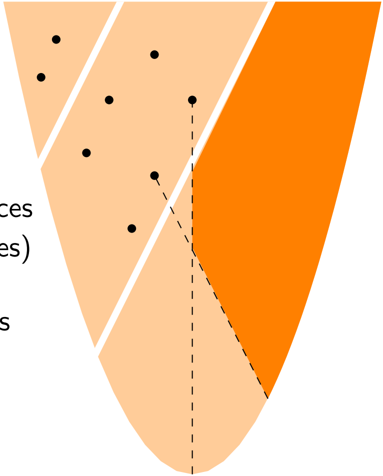
Assume he finished playing on the first slices
What is left of the next slice?



Reasoning with games: Slices



Slice X into disjoint subsets whose width is known (Antoine has a winning strategy)

Can Antoine combine his strategies on the slices into a winning strategy on X against Σ w(slices)

Assume he finished playing on the first slices
What is left of the next slice?

We need w(residual) = w(slice)

Reasoning with games: Slices

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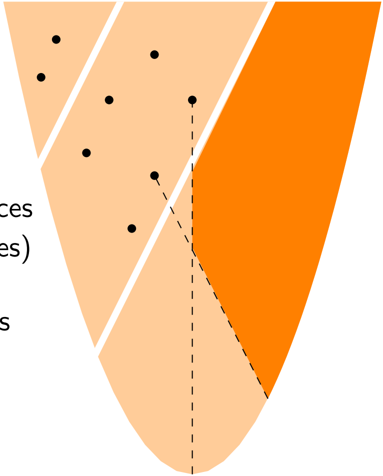
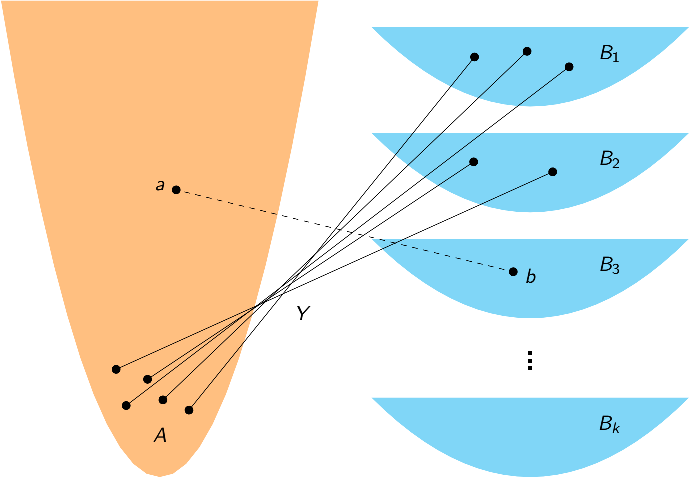
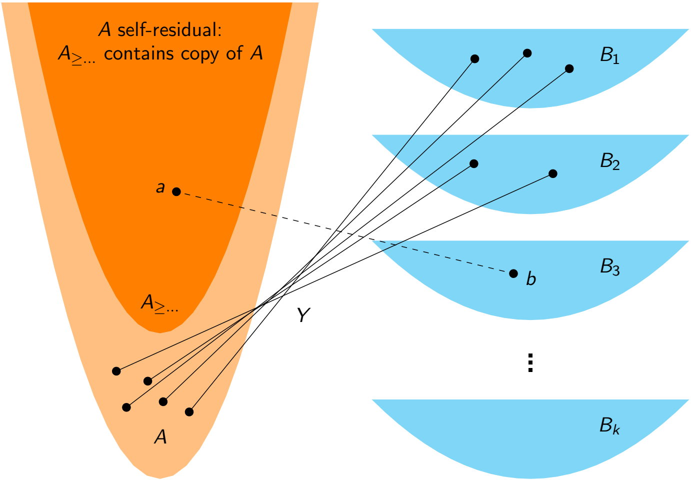


→ Quasi-incomparable subsets

Example: $\mathrm{w}(A \times (B + \cdots + B))$



Example: $\mathrm{w}(A \times (B + \cdots + B))$



♣ **If A self-residual**

Then $\mathrm{w}(A \times (B + \cdots + B)) = \mathrm{w}(A \times B) + \cdots + \mathrm{w}(A \times B)$

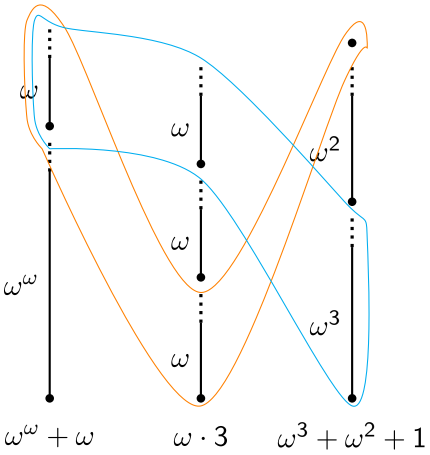
Study family of examples

Cartesian product of ordinals

A family to study width of CP

♣ **Computing** $w(\alpha_1 \times \cdots \times \alpha_n)$

- is functional
- $w(\alpha \times \beta)$ is known (Abraham (1987))
- Easy to slice into quasi-incomparable subsets $\omega^{\alpha_1} \times \cdots \times \omega^{\alpha_n}$
- New insight for CP of non-linear wqos



♣ Width of CP of n ordinals

$$\mathrm{w}(\alpha_1 \times \cdots \times \alpha_n) = \bigoplus_{\substack{s \in I_1 \times \cdots \times I_{k-1}, \\ \min s = 0}} \omega^{\eta\left(\alpha_{1,s(1)}, \ldots, \alpha_{k-1,s(k-1)}\right)} \otimes \left(\prod_{k \le i \le n} \alpha_i \right)$$

♦ When does one have $\mathrm{w} = \mathrm{o}$?

$\mathrm{w}(\alpha_1 \times \cdots \times \alpha_n) = \mathrm{o}(\alpha_1 \times \cdots \times \alpha_n)$ iff

- $\exists i$ s.t. $\alpha_i = \omega^\beta$
- $\exists j \neq k$ s.t. α_j and α_k are divisible by ω^ω

Cartesian product of ordinals

♦ New insight for the CP of non-linear wqos

Let $\mathrm{o}(A_i) = \alpha_i$. Then

$\mathrm{w}(\alpha_1 \times \cdots \times \alpha_n) \leq \mathrm{w}(A_1 \times \cdots \times A_n) \leq \mathrm{o}(A_1 \times \cdots \times A_n) = \mathrm{o}(\alpha_1 \times \cdots \times \alpha_n)$

♣ Translating conditions

$\mathrm{w}(A_1 \times \cdots \times A_n) = \mathrm{o}(A_1 \times \cdots \times A_n)$ if

- $\exists i$ s.t. $\mathrm{o}(A_i) = \omega^\beta$
- $\exists j \neq k$ s.t. $\mathrm{o}(A_j)$ and $\mathrm{o}(A_k)$ are divisible by ω^ω

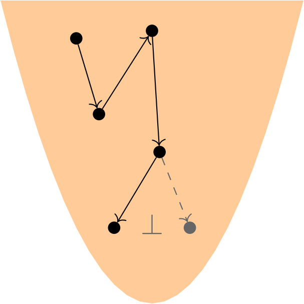
Third approach

Not functional in $\mathrm{o}, \mathrm{w}, \mathrm{h}$? Never mind! Let's find some new invariants

The fourth ordinal invariant

Definition (Friendly order type)

$$\mathrm{o}_{\perp}(X) \;=\; \text{rank of the tree of } \textit{open-ended} \text{ bad sequences}$$



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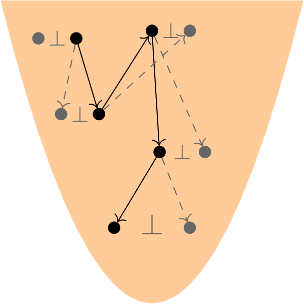


The fourth ordinal invariant

Theorem (Width of M°)

$\mathsf{w}(\mathsf{M}^\circ(X)) = \omega^{\circ_\perp(X)}$

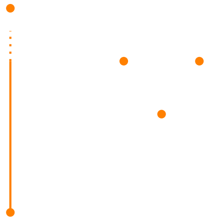
Space	o, h, w	$o_\perp$
		?
		?
		?

Theorem (Width of M°)

$w(M^\circ(X)) = \omega^{o_\perp(X)}$

♦ How to compute the fot?

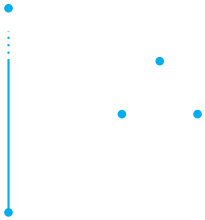
- Exists $X' \subseteq X$ such that $\mathrm{Bad}(X') \subseteq \mathrm{Bad}_\perp(X)$
- $\mathit{limit\_part}(o(\mathit{str}(X))) \leq o_\perp(X) \leq o(\mathit{str}(X))$ with $\mathit{str}(X) = \{\, x \in X \mid \exists y \in X, y \perp x \,\}$
- $w(X) - 1 \leq o_\perp(X)$
- if $w(A) = o(A)$ limit, then $o_\perp(X) = o(X)$
- $o_\perp(A \sqcup B) = o(A) \oplus o(B)$



$\nabla \cdot (\omega + 1)$

$o = \omega \cdot 3 + 2$

$max\_elt = 2$



$\Delta \cdot (\omega + 1)$

$o = \omega \cdot 3 + 1$

$max\_elt = 1$

Theorem (M.o.t. of the direct product)

$o(A) \cdot pred^k(o(B)) + o(A) \otimes k$ if $max\_elt(B) = k$

Conclusion

Space	M.O.T.	Height	Width
$A \sqcup B$	$\mathrm{o}(A) \oplus \mathrm{o}(B)$	$\max(\mathrm{h}(A), \mathrm{h}(B))$	$\mathrm{w}(A) \oplus \mathrm{w}(B)$
$A + B$	$\mathrm{o}(A) + \mathrm{o}(B)$	$\mathrm{h}(A) + \mathrm{h}(B)$	$\max(\mathrm{w}(A), \mathrm{w}(B))$
$A \times B$	$\mathrm{o}(A) \otimes \mathrm{o}(B)$	$\mathrm{h}(A) \hat{\oplus} \mathrm{h}(B)$	$\geq \mathrm{w}(\mathrm{o}(A) \times \mathrm{o}(B))$
$A \cdot B$	$\mathrm{o}(A) \cdot \mathit{pred}^k(\mathrm{o}(B)) + \mathrm{o}(A) \otimes k$ if $\mathit{max\_elt}(B) = k$	$\mathrm{h}(A) \cdot \mathrm{h}(B)$	$\mathrm{w}(A) \odot \mathrm{w}(B)$
A^*	$\omega^{\omega^{(\mathrm{o}(A)^{\pm})}}$	$\mathrm{h}^*(A)$	$\omega^{\omega^{(\mathrm{o}(A)^{\pm})}}$
$\mathrm{M}^\diamond(A)$	$\omega^{\widehat{\mathrm{o}(A)}}$	$\mathrm{h}^*(A)$	$\omega^{\widehat{\mathrm{o}(A)}-1}$
$\mathrm{M}^\circ(A)$	$\omega^{\mathrm{o}(A)}$	$\omega^{\mathrm{h}(A)}$	$\omega^{\mathrm{o}_\perp(A)}$
$\mathrm{P}_\mathrm{f}(A)$	$\leq 2^{\mathrm{o}(A)}$	$\leq 2^{\mathrm{h}(A)}$	$\geq 2^{\mathrm{w}(A)}$

Conclusion

♣ Measuring well quasi-orders

- is fun!
- Often not functional but. . . everyday-life wqos are well-behaved!
- Elementary family of wqos

$$E := \alpha \geq \omega^{\omega} \text{mult. indec.} \mid E_1 \sqcup E_2 \mid E_1 \times E_2 \mid \mathsf{M}^{\diamond}(E) \mid \mathsf{M}^{\circ}(E) \mid E^* \mid \mathsf{P_f}(E)$$

- Application in well-structured transition systems

Conclusion

♣ Measuring well quasi-orders

- is fun!
- Often not functional but... everyday-life wqos are well-behaved!
- Elementary family of wqos

$$E := \alpha \geq \omega^{\omega} \text{mult. indec.} \mid E_1 \sqcup E_2 \mid E_1 \times E_2 \mid \mathrm{M}^{\diamond}(E) \mid \mathrm{M}^{\circ}(E) \mid E^{*} \mid \mathrm{P_f}(E)$$

- Application in well-structured transition systems

♦ Open questions

- New invariants:
  - Computing the fot
  - Is there an invariant that would make CP and $\mathrm{P_f}$ functional?
- New operations: Infinite words, variants of trees, graph minor, ...

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On wqos

- I. Vialard, On the Width of the Cartesian Product of Ordinals, Order (2024).
- I. Vialard, Ordinal Measures of the Set of Finite Multisets, MFCS 2023.
- S. Abriola, S. Halfon, A. Lopez, S. Schmitz, Ph. Schnoebelen, I. Vialard, Measuring well quasi-ordered finitary powersets, soon to be submitted to MSCS.

On piecewise complexity and minimality index

- M. Praveen, Ph. Schnoebelen, I. Vialard, J. Veron, On the piecewise complexity of words and periodic words, SOFSEM 2024