Ordinal measures of the set of finite multisets

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♦ Well partial orders

- No infinite decreasing sequences, no infinite antichains
- No infinite bad sequences (Bad $= \forall i < j,\ x_i \not\leq x_j$)

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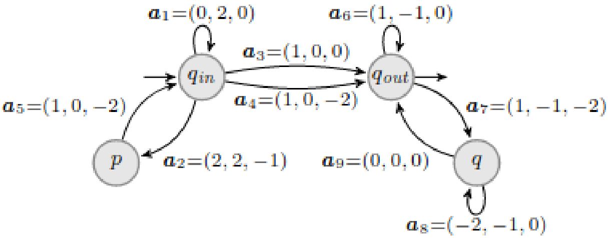
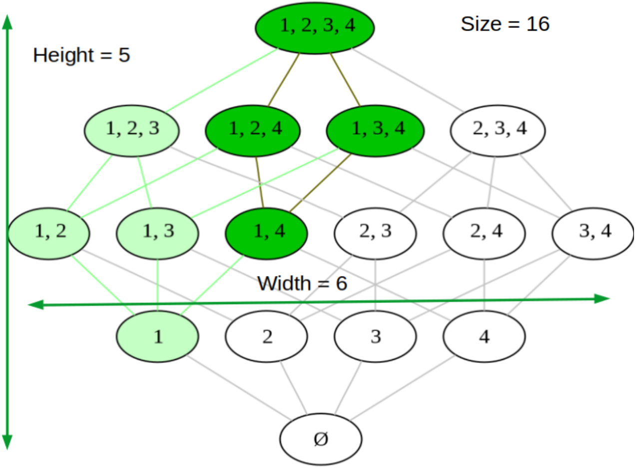
♣ Well-Structured Transition Systems

- Set of configurations = WPOs
- Ex : Counter Machine, VASS



♣ Intuitive notions of measure when finite



Ordinal invariants

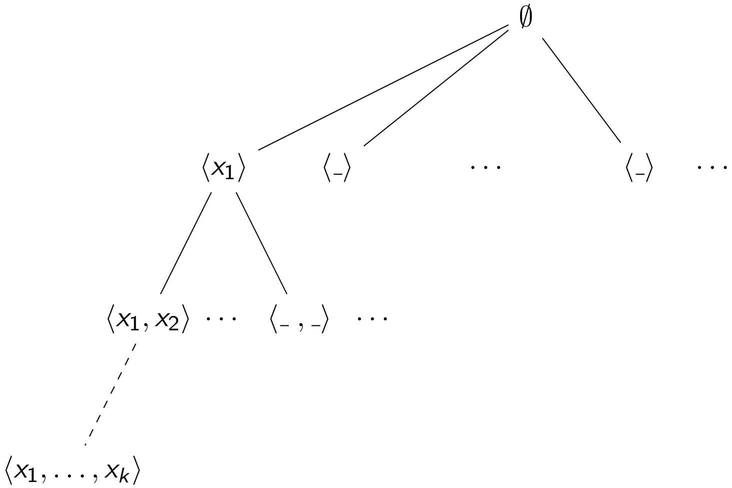
Definition (Maximal order type, Width and Height)

$$\left| \begin{array}{l} \boldsymbol{o}(X) \\ \boldsymbol{w}(X) \\ \boldsymbol{h}(X) \end{array} \right. = \underline{\textit{rank}} \text{ of the tree of } \begin{cases} \text{bad sequences} \\ \text{antichain sequences} \\ \text{decreasing sequences} \end{cases} \text{ in } X.$$

Ordinal invariants

♣ Rank of well-founded trees

(W.l.o.g. imagine it is a tree of decreasing sequences)



Ordinal invariants

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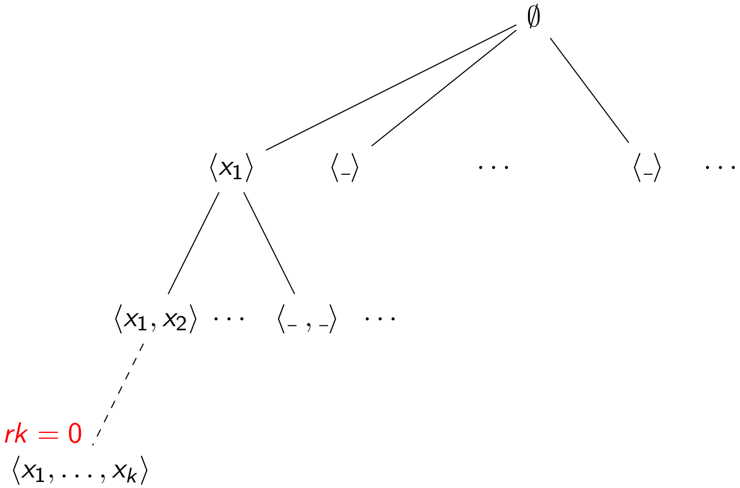

Ordinal invariants

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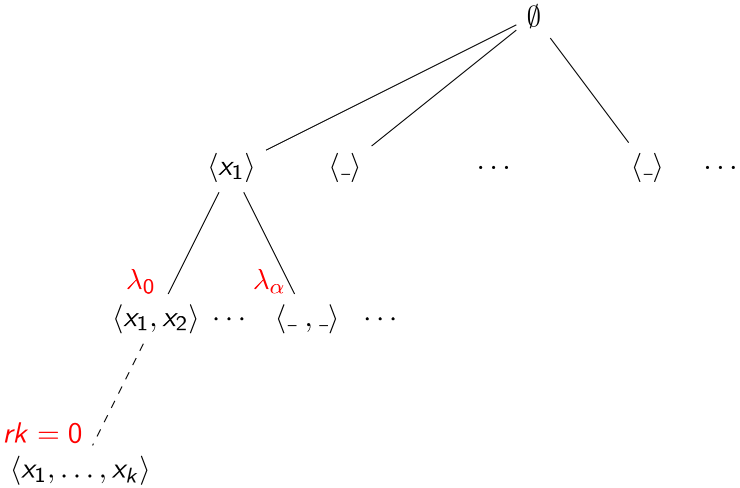

Ordinal invariants

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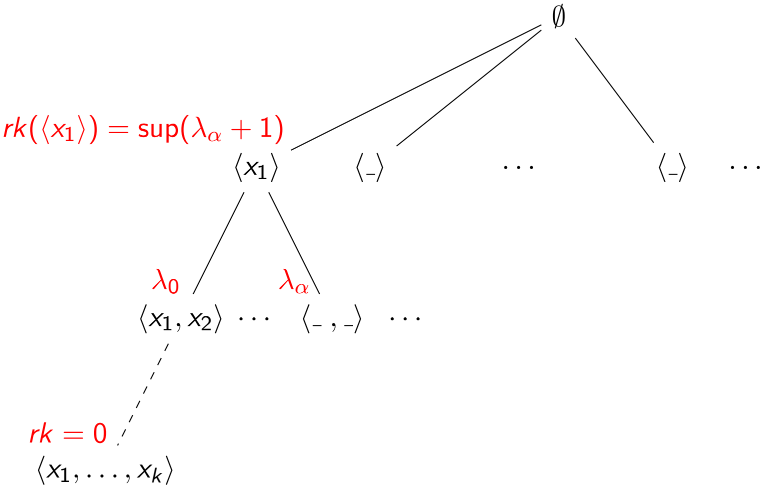

Ordinal invariants

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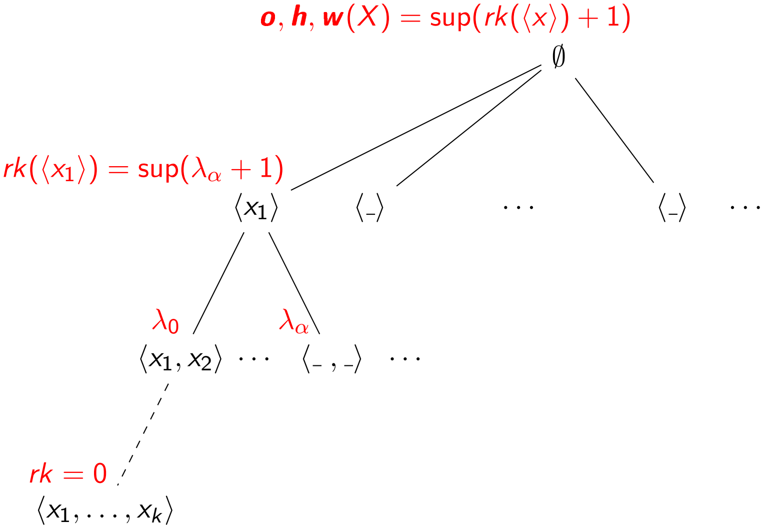

♣ Rank of well-founded trees

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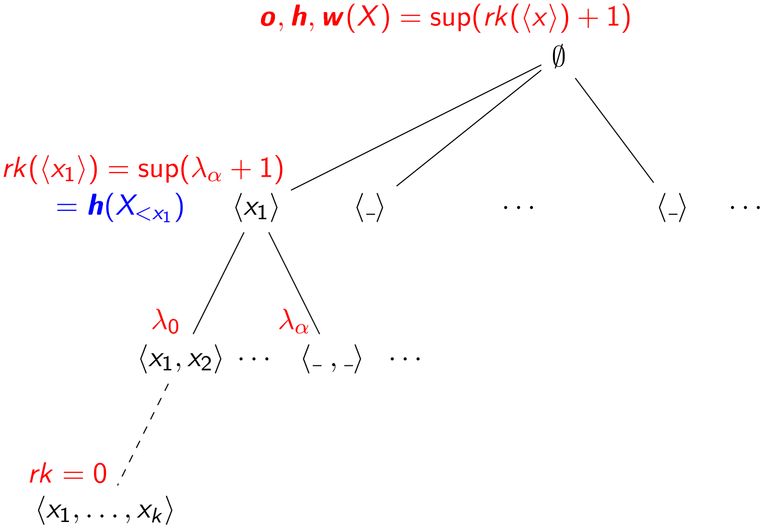
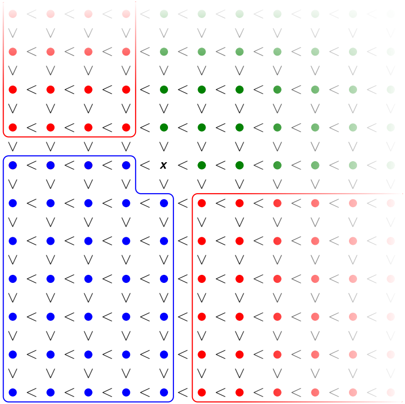


♦ **Descent equations**

$$\boldsymbol{o}(X) = \sup_{x \in X} \boldsymbol{o}(X_{\not\geq x}) + 1$$

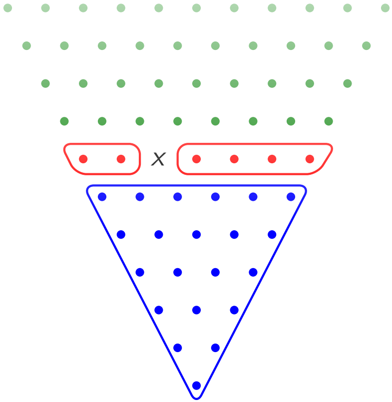
$$\boldsymbol{h}(X) = \sup_{x \in X} \boldsymbol{h}(X_{<x}) + 1$$

$$\boldsymbol{w}(X) = \sup_{x \in X} \boldsymbol{w}(X_{\perp x}) + 1$$



♣ **Ex: Residuals of $\mathbb{N} \times \mathbb{N}$**

Example : H



$$\boldsymbol{o}(H) = \boldsymbol{h}(H) = \boldsymbol{w}(H) = \omega$$

Two orderings on multiset

♦ **Multiset ordering M^o of a set (Ex: $\mathbb{N}$)**

$$\langle 30, 22, 22, 10\rangle >_o \langle 30, 22, 20, 20, 19, 10\rangle$$

- Used in the rewriting community
- Conserves linearity: $M^o(\alpha) \equiv \omega^\alpha$

$$\omega^{30} + \omega^{22} + \omega^{22} + \omega^{10} > \omega^{30} + \omega^{22} + \omega^{20} + \omega^{20} + \omega^{19} + \omega^{10}$$

Two orderings on multiset

♣ **Multiset embedding M^{e} of a set (Ex: $\mathbb{N}$)**

$$\langle 30, 22, 22, 10\rangle >_e \langle 30, 22, 20, 10\rangle$$
$$\langle 30, 22, 22, 10\rangle >_e \langle 30, 22, 10\rangle$$
$$\langle 30, 22, 22, 10\rangle \perp_e \langle 30, 22, 20, 20, 19, 10\rangle$$

- If $m \leq_e m'$ then $size(m) \leq size(m')$
- Does not conserve linearity: $\langle 1\rangle \perp_e \langle 0, 0\rangle$

♦ **If $m \leq_e m'$ then $m \leq_o m'$**

Ordinal invariants

♣ State of the art [1, 3]

Invariants	$\mathrm{M}^e(X)$	$M^o(X)$
Mot $\boldsymbol{o}$	$\omega^{\widehat{\boldsymbol{o}(X)}}$	$\omega^{\boldsymbol{o}(X)}$
Height $\boldsymbol{h}$	$\boldsymbol{h}^*(X)$	**?**
Width $\boldsymbol{w}$	**?**	**?**

Ordinal invariants

♣ State of the art [1, 3]

Invariants	$\mathrm{M}^{e}(X)$		$M^{o}(X)$
Mot $\boldsymbol{o}$	$\omega^{\widehat{\boldsymbol{o}(X)}}$	$\geq$	$\omega^{\boldsymbol{o}(X)}$
Height $\boldsymbol{h}$	$\boldsymbol{h}^{*}(X)$	$\leq$	?
Width $\boldsymbol{w}$	?	$\geq$	?

♦ **If** $m \leq_{e} m'$ **then** $m \leq_{o} m'$

Width of M^e

♣ **Kříž and Thomas's Lemma [2]**

$$\boldsymbol{o}(X) \leq \boldsymbol{w}(X) \otimes \boldsymbol{h}(X)$$

Invariants	$\mathsf{M}^e(X)$	$M^o(X)$
Mot $\boldsymbol{o}$	$\omega^{\widehat{\boldsymbol{o}(X)}}$	$\omega^{\boldsymbol{o}(X)}$
Height $\boldsymbol{h}$	$\boldsymbol{h}^*(X)$	**?**
Width $\boldsymbol{w}$	**?**	**?**

♣ Kříž and Thomas's Lemma [2]

$$\boldsymbol{o}(X) \leq \boldsymbol{w}(X) \otimes \boldsymbol{h}(X)$$

Theorem

$$\boldsymbol{w}(\mathsf{M}^{\mathsf{e}}(X)) = \omega^{\widehat{\boldsymbol{o}(X)}-1}$$

Invariants	$\mathsf{M}^{\mathsf{e}}(X)$	$M^o(X)$
Mot $\boldsymbol{o}$	$\omega^{\widehat{\boldsymbol{o}(X)}}$	$\omega^{\boldsymbol{o}(X)}$
Height $\boldsymbol{h}$	$\boldsymbol{h}^*(X)$	?
Width $\boldsymbol{w}$	$\omega^{\widehat{\boldsymbol{o}(X)}-1}$	?

Height of M^o

Theorem

$$\boldsymbol{h}(M^o(X)) = \omega^{\boldsymbol{h}(X)}$$

♣ **As expected**

- Consistent with $M^o(\alpha) \equiv \omega^\alpha$
- $\boldsymbol{h}(\mathrm{M^e}(X)) \leq \boldsymbol{h}(M^o(X))$
- Similar proof as $\boldsymbol{o}(M^o(X))$

Invariants	$\mathrm{M^e}(X)$	$M^o(X)$
Mot $\boldsymbol{o}$	$\omega^{\widehat{\boldsymbol{o}(X)}}$	$\omega^{\boldsymbol{o}(X)}$
Height $\boldsymbol{h}$	$\boldsymbol{h}^*(X)$	$\omega^{\boldsymbol{h}(X)}$
Width $\boldsymbol{w}$	$\omega^{\widehat{\boldsymbol{o}(X)}-1}$	**?**

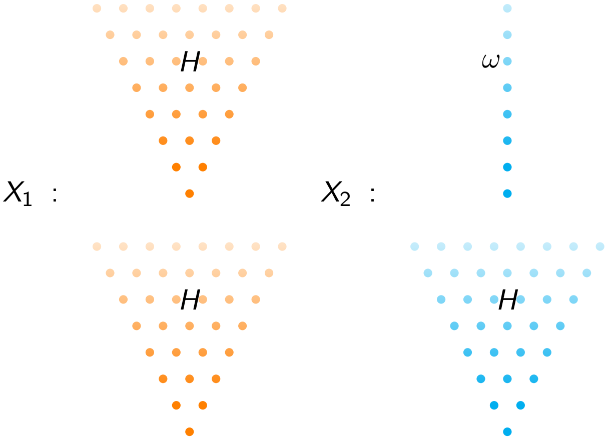
Width of M^o

Invariants	$\mathrm{M}^{\mathrm{e}}(X)$	$M^o(X)$
Mot $\boldsymbol{o}$	$\omega^{\widehat{\boldsymbol{o}(X)}}$	$\omega^{\boldsymbol{o}(X)}$
Height $\boldsymbol{h}$	$\boldsymbol{h}^*(X)$	$\omega^{\boldsymbol{h}(X)}$
Width $\boldsymbol{w}$	$\omega^{\widehat{\boldsymbol{o}(X)}-1}$	**?**

Width of M^o

Invariants	$\mathrm{M}^{\mathrm{e}}(X)$	$M^o(X)$
Mot $\boldsymbol{o}$	$\omega^{\widehat{\boldsymbol{o}(X)}}$	$\omega^{\boldsymbol{o}(X)}$
Height $\boldsymbol{h}$	$\boldsymbol{h}^*(X)$	$\omega^{\boldsymbol{h}(X)}$
Width $\boldsymbol{w}$	$\omega^{\widehat{\boldsymbol{o}(X)}-1}$	*Not functional*

Example



$$\boldsymbol{o}(X_i) = \omega + \omega \quad \boldsymbol{h}(X_i) = \omega + \omega \quad \boldsymbol{w}(X_i) = \omega$$

Lemma

$M^o(X+Y) = M^o(X) \cdot M^o(Y)$

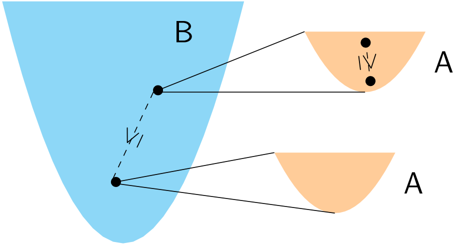
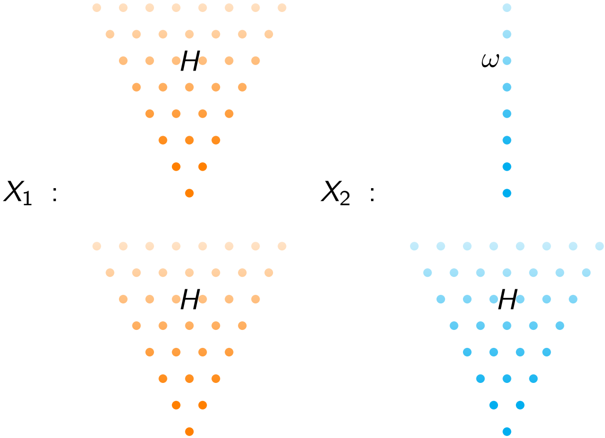


Figure 1: Lexicographic product $A \cdot B$, with $\boldsymbol{w}(A \cdot B) = \boldsymbol{w}(A) \odot \boldsymbol{w}(B)$

Back to example

$$\boldsymbol{w}(M^o(H)) = \omega^\omega \quad \boldsymbol{w}(M^o(\omega)) = \boldsymbol{w}(\omega^\omega) = 1$$

$$\boldsymbol{w}(M^o(X_1)) = \omega^\omega \odot \omega^\omega = \omega^{\omega \cdot 2} \neq \boldsymbol{w}(M^o(X_2)) = \omega^\omega \odot 1 = \omega^\omega$$



Definition (Friendly order type)

$$\boldsymbol{o}_{\perp}(X) \;=\; \text{rank of the tree of } \textit{open-ended} \text{ bad sequences}$$

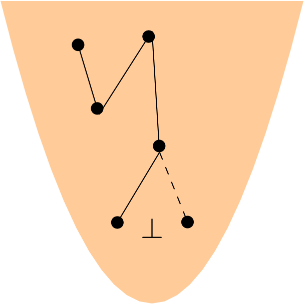


Figure 2: Open-ended bad sequence

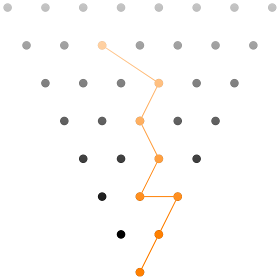
Theorem

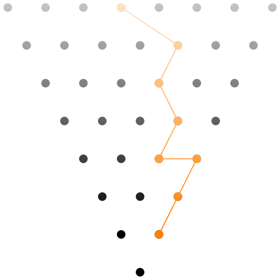
$$\boldsymbol{w}(M^o(X)) = \omega^{\boldsymbol{o}_\perp(X)}$$

Invariants	$\mathrm{M}^{\mathrm{e}}(X)$	$M^o(X)$
Mot $\boldsymbol{o}$	$\omega^{\widehat{\boldsymbol{o}(X)}}$	$\omega^{\boldsymbol{o}(X)}$
Height $\boldsymbol{h}$	$\boldsymbol{h}^*(X)$	$\omega^{\boldsymbol{h}(X)}$
Width $\boldsymbol{w}$	$\omega^{\widehat{\boldsymbol{o}(X)}-1}$	$\omega^{\boldsymbol{o}_\perp(X)}$

Examples

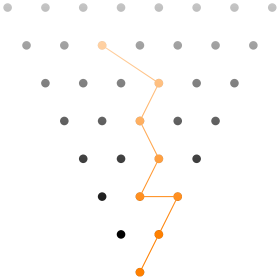
Bad sequence in H $\quad\rightarrow\quad$ Open-ended bad sequence

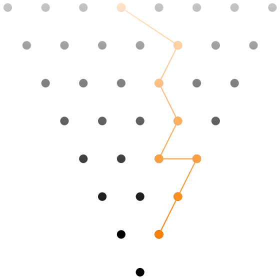




Examples

Bad sequence in H $\quad\rightarrow\quad$ Open-ended bad sequence

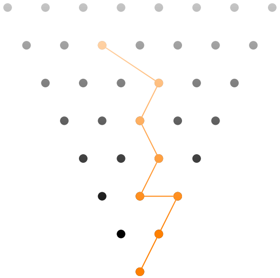


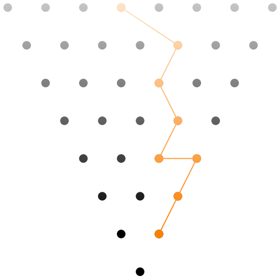


♣ $\boldsymbol{o}_\perp(H) = \boldsymbol{o}(H) = \omega$

Examples

Bad sequence in H $\rightarrow$ Open-ended bad sequence

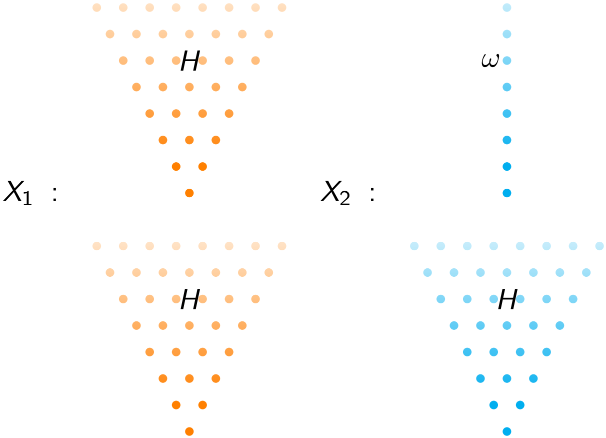




♣ $\boldsymbol{o}_{\perp}(H) = \boldsymbol{o}(H) = \omega$

♣ $\boldsymbol{o}_{\perp}(\alpha) = 0$

Back to X_1 and X_2



$$\boldsymbol{o}_\perp(X_1) = \boldsymbol{o}_\perp(H) + \boldsymbol{o}_\perp(H) = \omega \cdot 2 \qquad \boldsymbol{w}(M^o(X_1) = \omega^{\omega \cdot 2}$$

$$\boldsymbol{o}_\perp(X_2) = \boldsymbol{o}_\perp(H) + \boldsymbol{o}_\perp(\omega) = \omega \qquad \boldsymbol{w}(M^o(X_2) = \omega^{\omega}$$

New invariant = new questions

Invariants	$\mathrm{M}^e(X)$	$M^o(X)$
Mot $\boldsymbol{o}$	$\omega^{\widehat{\boldsymbol{o}(X)}}$	$\omega^{\boldsymbol{o}(X)}$
Height $\boldsymbol{h}$	$\boldsymbol{h}^*(X)$	$\omega^{\boldsymbol{h}(X)}$
Width $\boldsymbol{w}$	$\omega^{\widehat{\boldsymbol{o}(X)}-1}$	$\omega^{\boldsymbol{o}_\perp(X)}$
Fot $\boldsymbol{o}_\perp$	**?**	**?**

New invariant = new questions

Invariants	$\mathrm{M}^{e}(X)$	$M^{o}(X)$
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Fot $\boldsymbol{o}_{\perp}$	$\omega^{\widehat{\boldsymbol{o}(X)}}$	**?**

New invariant = new questions

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Height $\boldsymbol{h}$	$\boldsymbol{h}^*(X)$	$\omega^{\boldsymbol{h}(X)}$
Width $\boldsymbol{w}$	$\omega^{\widehat{\boldsymbol{o}(X)}-1}$	$\omega^{\boldsymbol{o}_\perp(X)}$
Fot $\boldsymbol{o}_\perp$	$\omega^{\widehat{\boldsymbol{o}(X)}}$	$\omega^{\boldsymbol{o}_\perp(X)} - 1$?

Thank you for listening !

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