Tropical algebra for piecewise complexity

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♣ Subword relation

$$RADAR \preceq AB\,RA\,C\,A\,D\,A\,B\,R\,A$$

♦ Simon's congruence

$u \equiv_m v$ iff $\forall w$ s.t. $|w| \leq m$, $w \preceq u \iff w \preceq v$

Ex. $NATIONALIST \equiv_2 ANTINATIONALIST$,

but $NATIONALIST \not\equiv_3 ANTINATIONALIST$: INO is a distinguisher.

Definition (Piecewise complexity)

$h(u)$ is the smallest k such that $u \equiv_k v$ implies $u = v$.

♣ **Example:** $h(abba) = 3$

$abba$ is the only word u such that $aba, bb \preceq u$ but $bab, aaa, bbb \not\preceq u$

[1] Sakarovitch J., Simon I. Subwords. Combinatorics on Words.

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Definition (Distance[1])

$$\begin{aligned}\delta(u, v) &= \max\{k \colon u \equiv_k v\} \\ &= |s| - 1 \text{ for } s \text{ a shortest distinguisher.}\end{aligned}$$

♣ **Example:** $\delta(abba, baba) = 2$

baa is a shortest distinguisher so $abba \equiv_2 baba$

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Definition (Distance[1])

$$\begin{aligned} \delta(u, v) &= \max\{k \colon u \equiv_k v\} \\ &= |s| - 1 \text{ for } s \text{ a shortest distinguisher.} \end{aligned}$$

♦ **Not intuitive**

$\delta(aaaaaa, b) = 0$ but $\delta(aaaaaa, aaaaaaa) = 6$

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$h = $ max of distance

Definition (Piecewise complexity)

$h(u)$ is the smallest k such that $u \equiv_k v$ implies $u = v$.

Definition (Distance[2])

$$\begin{aligned}\delta(u,v) &= \max\{k \colon u \equiv_k v\} \\ &= |s| - 1 \text{ for } s \text{ a shortest distinguisher.}\end{aligned}$$

♦ **Theorem**

$$h(u) = \max_{v \neq u} \delta(u,v) + 1$$

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$$\begin{aligned}h(u) &= \max_{v \neq u} \delta(u,v) + 1 \\ &= \max_{\substack{u = u_1 u_2 \\ a \in \Sigma}} \delta(u, u_1 a u_2) + 1\end{aligned}$$

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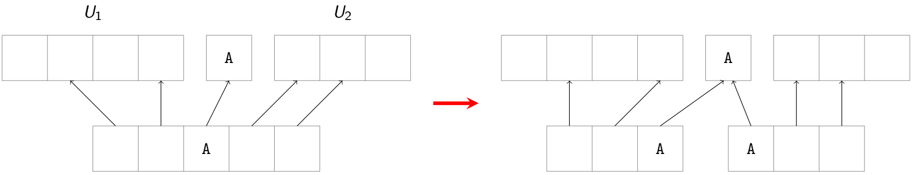
$h =$ max of distance

$$h(u) = \max_{\substack{u=u_1u_2 \\ a\in\Sigma}} \delta(u, u_1 a u_2) + 1$$

and

$$\begin{aligned}\delta(u, u_1 a u_2) &= \delta(u_1, u_1 a) + \delta(u_2, a u_2) \\ &= r(u_1, a) + \ell(a, u_2)^3\end{aligned}$$



[3]Simon, I. Piecewise testable events. In Automata Theory and Formal Languages (1975)

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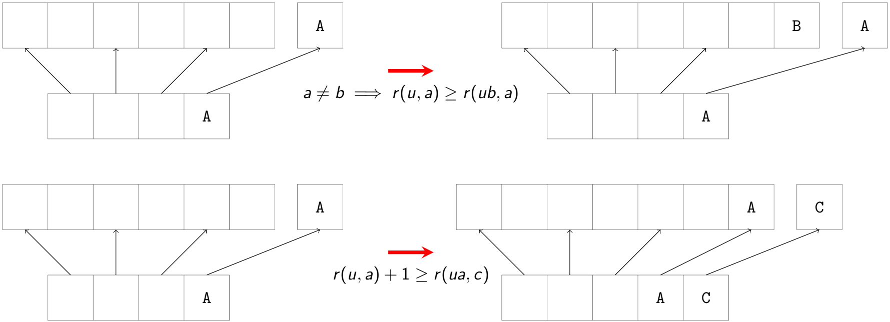
♦ Theorem [4]

$$r(\epsilon, a) = 0$$

$$r(ub, a) = \begin{cases} r(u, a) + 1 & \text{if } a = b, \\ \min\big(r(u, b) + 1,\ r(u, a)\big) & \text{if } a \neq b. \end{cases}$$



[4] Schnoebelen, Vialard, On the piecewise complexity of words. Acta Informatica (2025)

♦ Theorem [5]

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♣ Example

Let $u = \mathtt{ABBACCBCCABAABC}$ over $A = \{\mathtt{A}, \mathtt{B}, \mathtt{C}\}$. The r-table of u is

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
w		A	B	B	A	C	C	B	C	C	A	B	A	A	B	C
$r(i,\mathtt{A})$	0	1	1	1	2	1	1	1	1	1	2	2	3	4	4	3
$r(i,\mathtt{B})$	0	0	1	2	2	1	1	2	2	2	2	3	3	3	4	3
$r(i,\mathtt{C})$	0	0	0	0	0	1	2	2	3	4	2	2	2	2	2	3

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Definition

The r-table R_u: $R_u(i,j) = r(u_j, a_i)$ where $u_j = u[1,j]$

The ℓ-table L_u: $L_u(i,j) = \ell(a_i, u'_j)$ where $u'_j = u[j+1, |u|]$

♣ Example

Here are R_u and L_u for $u = \texttt{CBBCAC}$, assuming $\Sigma = \{\texttt{A}, \texttt{B}, \texttt{C}\}$:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 & \boxed{2} & 1 & 2 \end{pmatrix} \qquad \begin{array}{l} r\big(u[,4], \texttt{C}\big) = 2 \\ \ell\big(\texttt{C}, u[4,]\big) = 1 \end{array} \qquad \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 2 & \boxed{1} & 1 & 0 \end{pmatrix}$$

♦ Scanning $(R+L)_u$ for maximum: Complexity in $|u| \cdot |\Sigma|$ [6]

[6] Schnoebelen, Vialard, On the piecewise complexity of words. Acta Informatica (2025)

Our motivation

Definition

SLP = deterministic production rules $X \to YZ, X \to \mathtt{A}$, generating 1 word

♦ **Goal: compute $h(u)$ for u given as a SLP**

. . . in size of SLP (instead of size of u)

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Ex: $h(ab) = h(ba) = 2$ but $h(abba) = 3$

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Convexity: $h(u), h(v) \leq h(uv) \leq h(u) + h(v) - 1$

We need a *signature* $S(u)$

- with enough information to compute $h(u)$
- such that $S(uv)$ can be computed from $S(u)$ and $S(v)$

Now tropical algebra appears!

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$r(i,\mathtt{A})$	0	1	1	1	2	1	1	1	1	1	2	2	3	4	4	3
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$r(i,\mathtt{C})$	0	0	0	0	0	1	2	2	3	4	2	2	2	2	2	3

$$\begin{pmatrix} 2 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 1+1 \\ \min 2, 1+1 \\ \min 4, 1+1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 4 \end{pmatrix} \cdot \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ * & 0 & * \\ * & * & 0 \end{pmatrix}}_{M_{\mathtt{A}}}$$

Now tropical algebra appears!

♦ Min-plus algebra

- Just like Max-plus
- $*$ denotes $+\infty$

$$\begin{pmatrix} 2 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 1+1 \\ \min 2, 1+1 \\ \min 4, 1+1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 4 \end{pmatrix} \cdot \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ * & 0 & * \\ * & * & 0 \end{pmatrix}}_{M_{\mathtt{A}}}$$

♣ Introduced by Simon

- $\exists m$? s.t. $A^m = A^\star$
- Finding minimal paths in graph

Now tropical algebra appears!

- **Min-plus algebra**
  - Just like Max-plus
  - $*$ denotes $+\infty$

Definition

For $u = u_1 \dots u_m$, $M_u = M_{u_1} \cdot \dots \cdot M_{u_n}$.

$$\begin{pmatrix} 2 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 1+1 \\ \min 2, 1+1 \\ \min 4, 1+1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 4 \end{pmatrix} \cdot \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ * & 0 & * \\ * & * & 0 \end{pmatrix}}_{M_{\mathtt{A}}}$$

$$= \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \cdot M_u$$

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
w		A	B	B	A	C	C	B	C	C	A	B	A	A	B	C
$r(i, \mathtt{A})$	0	1	1	1	2	1	1	1	1	1	2	2	3	4	4	3
$r(i, \mathtt{B})$	0	0	1	2	2	1	1	2	2	2	2	3	3	3	4	3
$r(i, \mathtt{C})$	0	0	0	0	0	1	2	2	3	4	2	2	2	2	2	3

Definition

The matrix r-table F_u:

$$r(i,a_j) = (0\dots0)\cdot M_{u_i}(\cdot,j) = (0\dots0)\cdot F_u(i,j)$$

♣ Example ($u = \texttt{CBBCAC}$)

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 & 2 & 1 & 2 \end{pmatrix} \to \left| \begin{matrix} \begin{pmatrix} 0 \\ * \\ * \end{pmatrix} & \begin{pmatrix} 0 \\ * \\ 1 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} & \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} & \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \\ \begin{pmatrix} * \\ 0 \\ * \end{pmatrix} & \begin{pmatrix} * \\ 0 \\ 1 \end{pmatrix} & \begin{pmatrix} * \\ 1 \\ 2 \end{pmatrix} & \begin{pmatrix} * \\ 2 \\ 3 \end{pmatrix} & \begin{pmatrix} * \\ 2 \\ 2 \end{pmatrix} & \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} & \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \\ \begin{pmatrix} * \\ * \\ 0 \end{pmatrix} & \begin{pmatrix} * \\ * \\ 1 \end{pmatrix} & \begin{pmatrix} * \\ 1 \\ 1 \end{pmatrix} & \begin{pmatrix} * \\ 1 \\ 1 \end{pmatrix} & \begin{pmatrix} * \\ 2 \\ 2 \end{pmatrix} & \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} & \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix} \end{matrix} \right|$$

Now combining the new r and ℓ-tables

♣ Example ($u = \texttt{CBBCAC}$)

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 & \boxed{2} & 1 & 2 \end{pmatrix} \qquad \begin{matrix} r(u[,4],\texttt{C}) = 2 \\ \ell(\texttt{C}, u[4,]) = 1 \end{matrix} \qquad \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 2 & \boxed{1} & 1 & 0 \end{pmatrix}$$

$$\left|\begin{matrix} \binom{0}{*\,*} \end{matrix}\right.$$

$$\left|\begin{matrix} \begin{pmatrix} 0\\ *\\ * \end{pmatrix} & \begin{pmatrix} 0\\ *\\ 1 \end{pmatrix} & \begin{pmatrix} 0\\ 1\\ 1 \end{pmatrix} & \begin{pmatrix} 0\\ 1\\ 1 \end{pmatrix} & \begin{pmatrix} 0\\ 1\\ 1 \end{pmatrix} & \begin{pmatrix} 1\\ 2\\ 2 \end{pmatrix} & \begin{pmatrix} 1\\ 2\\ 2 \end{pmatrix} \\ \begin{pmatrix} *\\ 0\\ * \end{pmatrix} & \begin{pmatrix} *\\ 0\\ 1 \end{pmatrix} & \begin{pmatrix} *\\ 1\\ 2 \end{pmatrix} & \begin{pmatrix} *\\ 2\\ 3 \end{pmatrix} & \begin{pmatrix} *\\ 2\\ 2 \end{pmatrix} & \begin{pmatrix} 1\\ 2\\ 2 \end{pmatrix} & \begin{pmatrix} 1\\ 2\\ 2 \end{pmatrix} \\ \begin{pmatrix} *\\ *\\ 0 \end{pmatrix} & \begin{pmatrix} *\\ *\\ 1 \end{pmatrix} & \begin{pmatrix} *\\ 1\\ 1 \end{pmatrix} & \begin{pmatrix} *\\ 1\\ 1 \end{pmatrix} & \begin{pmatrix} *\\ 2\\ 2 \end{pmatrix} & \boxed{\begin{pmatrix} 1\\ 2\\ 2 \end{pmatrix}} & \begin{pmatrix} 2\\ 3\\ 3 \end{pmatrix} \end{matrix}\right| \qquad \left|\begin{matrix} \begin{pmatrix} 1\\ 1\\ 2 \end{pmatrix} & \begin{pmatrix} 1\\ 1\\ 2 \end{pmatrix} & \begin{pmatrix} 1\\ 1\\ 2 \end{pmatrix} & \begin{pmatrix} 1\\ *\\ 2 \end{pmatrix} & \begin{pmatrix} 1\\ *\\ 2 \end{pmatrix} & \begin{pmatrix} 0\\ *\\ 1 \end{pmatrix} & \begin{pmatrix} 0\\ *\\ * \end{pmatrix} \\ \begin{pmatrix} 3\\ 2\\ 3 \end{pmatrix} & \begin{pmatrix} 3\\ 2\\ 3 \end{pmatrix} & \begin{pmatrix} 2\\ 1\\ 2 \end{pmatrix} & \begin{pmatrix} 1\\ 0\\ 1 \end{pmatrix} & \begin{pmatrix} 1\\ 0\\ 1 \end{pmatrix} & \begin{pmatrix} *\\ 0\\ 1 \end{pmatrix} & \begin{pmatrix} *\\ 0\\ * \end{pmatrix} \\ \begin{pmatrix} 3\\ 2\\ 3 \end{pmatrix} & \begin{pmatrix} 2\\ 1\\ 2 \end{pmatrix} & \begin{pmatrix} 2\\ 1\\ 2 \end{pmatrix} & \begin{pmatrix} 2\\ *\\ 2 \end{pmatrix} & \boxed{\begin{pmatrix} 1\\ *\\ 1 \end{pmatrix}} & \begin{pmatrix} *\\ *\\ 1 \end{pmatrix} & \begin{pmatrix} *\\ *\\ 0 \end{pmatrix} \end{matrix}\right|$$

$$(*\ 2\ 2) \otimes (1\ *\ 1) = \begin{pmatrix} * & * & * \\ 3 & * & 3 \\ 3 & * & 3 \end{pmatrix}$$

Our signature

♦ Signature $S(u)$

- M_u, M_{u^R}
- H_u the maximum elements of $F_u \otimes G_u$

♣ Compute $h(u)$ from $S(u)$

- $r(u_j, a_i) = (0 \ldots 0) \cdot F_u(i,j) = \min F_u(i,j)$
- $r(u_j, a_i) + \ell(a_i, u') = \min(F_u \otimes G_u)(i,j)$ for $u = u_j u'$

Hence

$$
\begin{aligned}
max_{i,j} r(u_j, a_i) + \ell(a_i, u') + 1 &= \max_{i,j} \min(F_u \otimes G_u)(i,j) + 1 \\
&= \max_{M \in H_u} \min(F_u \otimes G_u)(i,j) + 1
\end{aligned}
$$

Concatenating tables

♣ **Example:** F_{uv} **for** $u = \texttt{CBB}$, $v = \texttt{CAC}$

$$
\begin{array}{c}
\begin{array}{ccc} \texttt{C} & \texttt{B} & \texttt{B} \end{array} \\
\left|\begin{matrix}
\begin{pmatrix}0\\\ast\\\ast\end{pmatrix} & \begin{pmatrix}0\\\ast\\1\end{pmatrix} & \begin{pmatrix}0\\1\\1\end{pmatrix} & \begin{pmatrix}0\\1\\1\end{pmatrix} \\
\begin{pmatrix}\ast\\0\\\ast\end{pmatrix} & \begin{pmatrix}\ast\\0\\1\end{pmatrix} & \begin{pmatrix}\ast\\1\\2\end{pmatrix} & \begin{pmatrix}\ast\\2\\3\end{pmatrix} \\
\begin{pmatrix}\ast\\\ast\\0\end{pmatrix} & \begin{pmatrix}\ast\\\ast\\1\end{pmatrix} & \begin{pmatrix}\ast\\1\\1\end{pmatrix} & \begin{pmatrix}\ast\\1\\1\end{pmatrix}
\end{matrix}\right|
\end{array}
::
\begin{array}{c}
\begin{array}{ccc} \texttt{C} & \texttt{A} & \texttt{C} \end{array} \\
\left|\begin{matrix}
\begin{pmatrix}0\\\ast\\\ast\end{pmatrix} & \begin{pmatrix}0\\\ast\\1\end{pmatrix} & \begin{pmatrix}1\\\ast\\2\end{pmatrix} & \begin{pmatrix}1\\\ast\\2\end{pmatrix} \\
\begin{pmatrix}\ast\\0\\\ast\end{pmatrix} & \begin{pmatrix}\ast\\0\\1\end{pmatrix} & \begin{pmatrix}1\\0\\1\end{pmatrix} & \begin{pmatrix}1\\0\\1\end{pmatrix} \\
\begin{pmatrix}\ast\\\ast\\0\end{pmatrix} & \begin{pmatrix}\ast\\\ast\\1\end{pmatrix} & \begin{pmatrix}1\\\ast\\1\end{pmatrix} & \begin{pmatrix}2\\\ast\\2\end{pmatrix}
\end{matrix}\right|
\end{array}
$$

$$
= \begin{array}{c}
\left|\begin{matrix}
\begin{pmatrix}0\\\ast\\\ast\end{pmatrix} & \begin{pmatrix}0\\\ast\\1\end{pmatrix} & \begin{pmatrix}0\\1\\1\end{pmatrix} & \begin{pmatrix}0\\1\\1\end{pmatrix} & M_u\begin{pmatrix}0\\\ast\\1\end{pmatrix} & M_u\begin{pmatrix}1\\\ast\\2\end{pmatrix} & M_u\begin{pmatrix}1\\\ast\\2\end{pmatrix} \\
\begin{pmatrix}\ast\\0\\\ast\end{pmatrix} & \begin{pmatrix}\ast\\0\\1\end{pmatrix} & \begin{pmatrix}\ast\\1\\2\end{pmatrix} & \begin{pmatrix}\ast\\2\\3\end{pmatrix} & M_u\begin{pmatrix}\ast\\0\\1\end{pmatrix} & M_u\begin{pmatrix}1\\0\\1\end{pmatrix} & M_u\begin{pmatrix}1\\0\\1\end{pmatrix} \\
\begin{pmatrix}\ast\\\ast\\0\end{pmatrix} & \begin{pmatrix}\ast\\\ast\\1\end{pmatrix} & \begin{pmatrix}\ast\\1\\1\end{pmatrix} & \begin{pmatrix}\ast\\1\\1\end{pmatrix} & M_u\begin{pmatrix}\ast\\\ast\\1\end{pmatrix} & M_u\begin{pmatrix}1\\\ast\\1\end{pmatrix} & M_u\begin{pmatrix}2\\\ast\\2\end{pmatrix}
\end{matrix}\right| \\
\begin{array}{cccccc} \texttt{C} & \texttt{B} & \texttt{B} & \texttt{C} & \texttt{A} & \texttt{C} \end{array}
\end{array}
$$

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$$
\begin{array}{c}
\begin{array}{ccccccc} & \mathtt{C} & & \mathtt{B} & & \mathtt{B} & \end{array}
\\
\left|\begin{array}{cccc}
\begin{pmatrix}0\\ *\\ *\end{pmatrix} & \begin{pmatrix}0\\ *\\ 1\end{pmatrix} & \begin{pmatrix}0\\ 1\\ 1\end{pmatrix} & \begin{pmatrix}0\\ 1\\ 1\end{pmatrix} \\
\begin{pmatrix}*\\ 0\\ *\end{pmatrix} & \begin{pmatrix}*\\ 0\\ 1\end{pmatrix} & \begin{pmatrix}*\\ 1\\ 2\end{pmatrix} & \begin{pmatrix}*\\ 2\\ 3\end{pmatrix} \\
\begin{pmatrix}*\\ *\\ 0\end{pmatrix} & \begin{pmatrix}*\\ *\\ 1\end{pmatrix} & \begin{pmatrix}*\\ 1\\ 1\end{pmatrix} & \begin{pmatrix}*\\ 1\\ 1\end{pmatrix}
\end{array}\right|
\end{array}
::
\begin{array}{c}
\begin{array}{ccccccc} & \mathtt{C} & & \mathtt{A} & & \mathtt{C} & \end{array}
\\
\left|\begin{array}{cccc}
\begin{pmatrix}0\\ *\\ *\end{pmatrix} & \begin{pmatrix}0\\ *\\ 1\end{pmatrix} & \begin{pmatrix}1\\ *\\ 2\end{pmatrix} & \begin{pmatrix}1\\ *\\ 2\end{pmatrix} \\
\begin{pmatrix}*\\ 0\\ *\end{pmatrix} & \begin{pmatrix}*\\ 0\\ 1\end{pmatrix} & \begin{pmatrix}1\\ 0\\ 1\end{pmatrix} & \begin{pmatrix}1\\ 0\\ 1\end{pmatrix} \\
\begin{pmatrix}*\\ *\\ 0\end{pmatrix} & \begin{pmatrix}*\\ *\\ 1\end{pmatrix} & \begin{pmatrix}1\\ *\\ 1\end{pmatrix} & \begin{pmatrix}2\\ *\\ 2\end{pmatrix}
\end{array}\right|
\end{array}
$$

$$
= \begin{array}{c}
\left|\begin{array}{ccccccc}
\begin{pmatrix}0\\ *\\ *\end{pmatrix} & \begin{pmatrix}0\\ *\\ 1\end{pmatrix} & \begin{pmatrix}0\\ 1\\ 1\end{pmatrix} & \begin{pmatrix}0\\ 1\\ 1\end{pmatrix} & M_u\begin{pmatrix}0\\ *\\ 1\end{pmatrix} & M_u\begin{pmatrix}1\\ *\\ 2\end{pmatrix} & M_u\begin{pmatrix}1\\ *\\ 2\end{pmatrix} \\
\begin{pmatrix}*\\ 0\\ *\end{pmatrix} & \begin{pmatrix}*\\ 0\\ 1\end{pmatrix} & \begin{pmatrix}*\\ 1\\ 2\end{pmatrix} & \begin{pmatrix}*\\ 2\\ 3\end{pmatrix} & M_u\begin{pmatrix}*\\ 0\\ 1\end{pmatrix} & M_u\begin{pmatrix}1\\ 0\\ 1\end{pmatrix} & M_u\begin{pmatrix}1\\ 0\\ 1\end{pmatrix} \\
\begin{pmatrix}*\\ *\\ 0\end{pmatrix} & \begin{pmatrix}*\\ *\\ 1\end{pmatrix} & \begin{pmatrix}*\\ 1\\ 1\end{pmatrix} & \begin{pmatrix}*\\ 1\\ 1\end{pmatrix} & M_u\begin{pmatrix}*\\ *\\ 1\end{pmatrix} & M_u\begin{pmatrix}1\\ *\\ 1\end{pmatrix} & M_u\begin{pmatrix}2\\ *\\ 2\end{pmatrix}
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\end{array}
$$

Concatenating tables

♦ **Signature $S(u)$**

- M_u, M_{u^R}
- H_u the maximum elements of $F_u \otimes G_u$

♣ **Compute $S(uv)$ from $S(u)$ and $S(v)$**

- $F_{uv}(i,j) = F_u(i,j)$ or $M_u \cdot F_v(i,j)$
- $G_{uv}(i,j) = G_u(i,j) \cdot M_{v^R}$ or $F_v(i,j)$

Hence $H_{uv} \subseteq M_u \cdot H_v \cup H_u \cdot M_{v^R}$

Size of our signature

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NO: $H_u \leq |\Sigma| \cdot (2|\Sigma| + 1)$

- Bound on the largest antichain if $F_u \otimes G_u$

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- H_u the set of maximum elements of $F_u \otimes G_u$

♣ **Does $|H_u|$ explodes when concatenating?**

NO: $H_u \leq |\Sigma| \cdot (2|\Sigma| + 1)$

- Bound on the largest antichain if $F_u \otimes G_u$
- Almost tight: $|H_{u_k}| \geq k \cdot (k+1)$ for

$$u_k = a_1 \ldots a_k a_k a_{k-1} a_k \ldots a_2 \ldots a_k a_1 a_k \ldots a_2 \ldots a_k a_{k-1} a_k a_k \ldots a_1$$

Other use of tropical algebra

Definition

Minimality index $\rho(u)$ is the smallest k such that $u \equiv_k v$ implies $u = v$ ...

... for all $v \preceq u$

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♦ Theorem

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- $\rho(u) \leq h(u) - 1$ ($=$ if $|\Sigma| = 2$)

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♣ Tweak our algorithm for ρ:

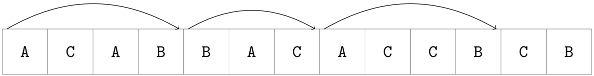
Bounds on $|H_u|$?

- $|H_u| \leq |\Sigma| \cdot (|\Sigma| + 2)$.
- Might not be tight

Other use of tropical algebra

Definition

Universality index $\iota(u) = \max\{k : |v| \leq k \implies v \preceq u\}$



Ex: $\iota(u) = 3$

♣ **Example** $(u = \texttt{ACABBACACCBCB})$

Theorem

$\iota(u) = \min M_u$

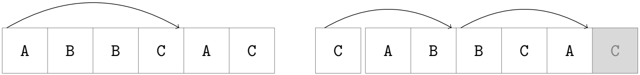
$$M_u = \begin{pmatrix} 4 & 5 & 5 \\ 3 & 4 & 4 \\ 3 & 4 & 4 \end{pmatrix}$$

Other use of tropical algebra

Definition

Circular universality index

$$\zeta(u) = \max_{u=u_1 u_2} \iota(u_2 u_1)$$



Ex: $\iota(u) = 1$ but $\zeta(u) = 2$

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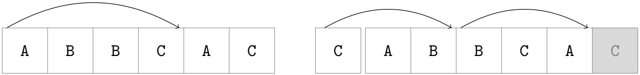
Circular universality index

$$\zeta(u) = \max_{u = u_1 u_2} \iota(u_2 u_1)$$



Ex: $\iota(u) = 1$ but $\zeta(u) = 2$

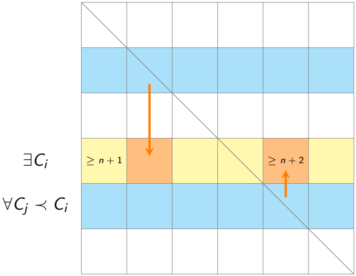
♣ Can you see the pattern ?

$$M_{\texttt{ABBCAC}} = \begin{pmatrix} 2 & 2 & 3 \\ 2 & 2 & 3 \\ 2 & 1 & 2 \end{pmatrix}$$

♦ Theorem

If $\iota(u) = n$, then $\zeta(u) = n + 1$ iff:

Otherwise $\zeta(u) = n$



♣ Can't guess this pattern [7]

$$M_{\mathtt{ABBCAC}} = \begin{pmatrix} 2 & 2 & 3 \\ 2 & 2 & 3 \\ 2 & 1 & 2 \end{pmatrix}$$

[7] Schnoebelen, Veron,(2023). On Arch Factorization and Subword Universality for Words and Compressed Words, WORDS 2023.

Conclusion

♣ **Tropical algebra is a natural tool to study these kind of measures**

- Easy to implement data structures
- M_u is a gold mine: we can extract $\iota(u), \zeta(u), \ldots$
- … and how u "behaves" when concatenated to other words

♦ **Some open questions**

- Tight bounds on H_u for $\rho(u)$
- Bounds on the relation between $|u|$ and $h(u)$
- Same question for $\rho(u)$