Elementary, Dr Powerset !

A sequel to : Silence of the Powersets

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What to expect

♣ Previously in Silence of the Powersets

- The ordinal invariants of $\mathcal{P}_{\mathsf{fin}}$ are not functional...
- ... but can be bounded !

♦ In this episode

- Ordinal invariants of a family of *elementary* WQOs
- Bonus: some exciting results on the width of the cartesian product

In the previous episode...

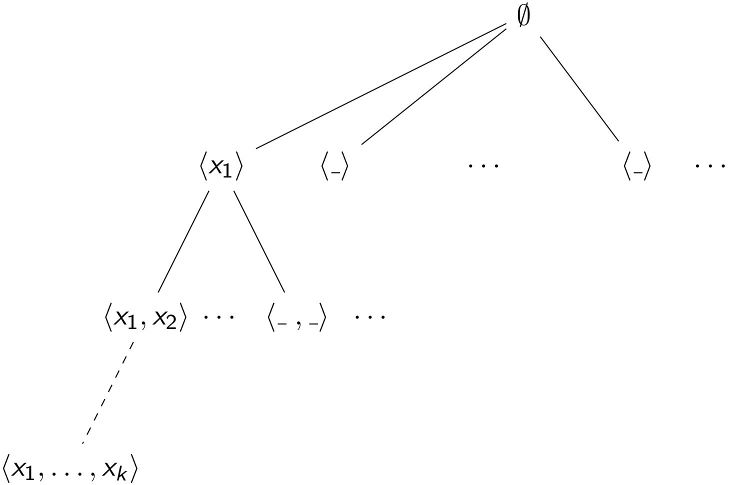
Ordinal invariants

Definition (Maximal order type, Width and Height)

$$\left|\begin{array}{l}\boldsymbol{o}(X)\\ \boldsymbol{w}(X)\\ \boldsymbol{h}(X)\end{array}\right. = \text{rank of root in the tree of} \begin{cases}\text{bad sequences}\\ \text{antichains}\\ \text{decreasing sequences}\end{cases} \text{in } X.$$

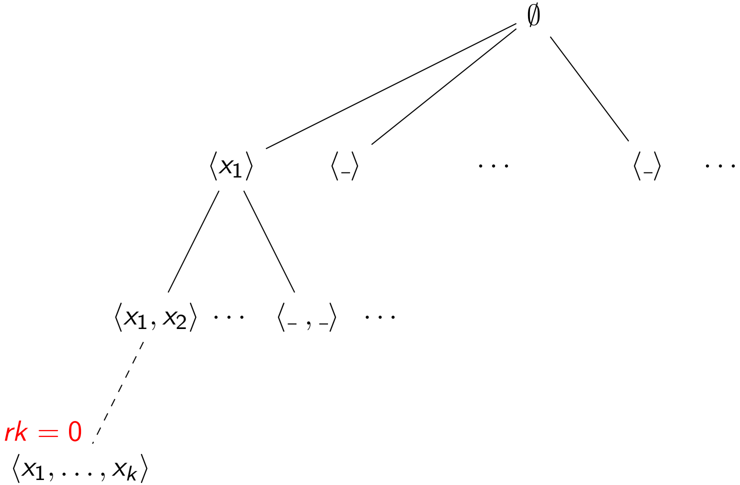
Ordinal invariants

♣ Rank of well-founded trees



Ordinal invariants

♣ Rank of well-founded trees



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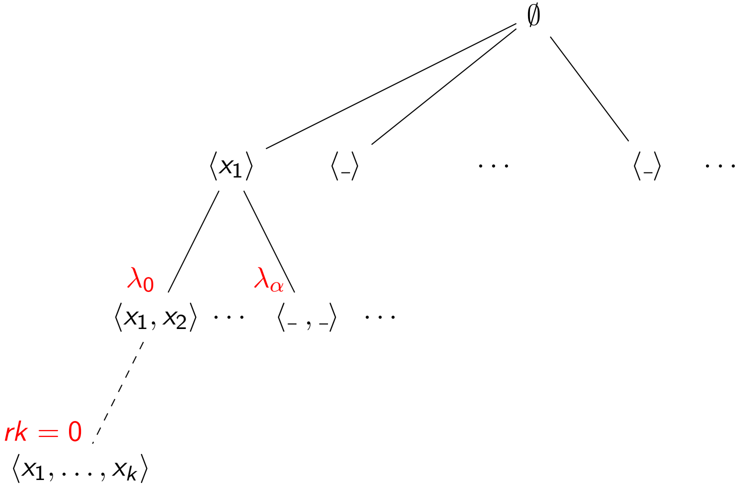

Ordinal invariants

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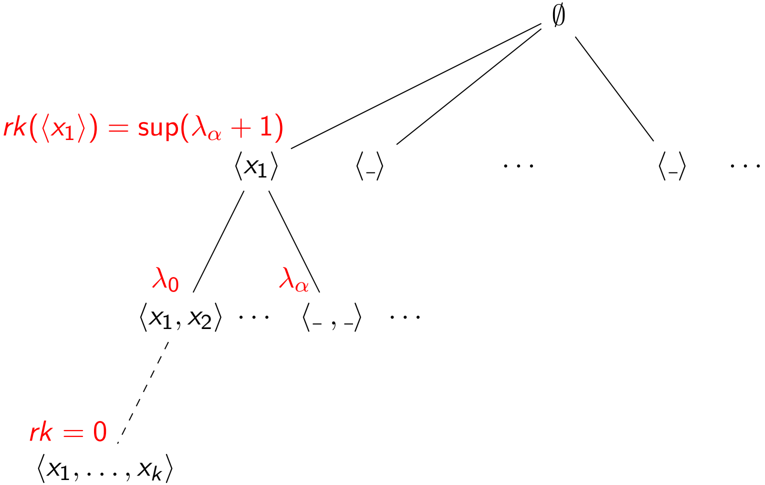

Ordinal invariants

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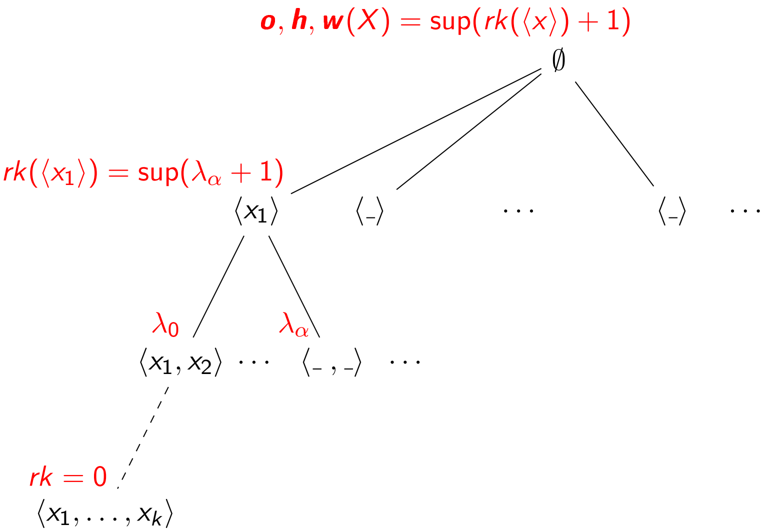
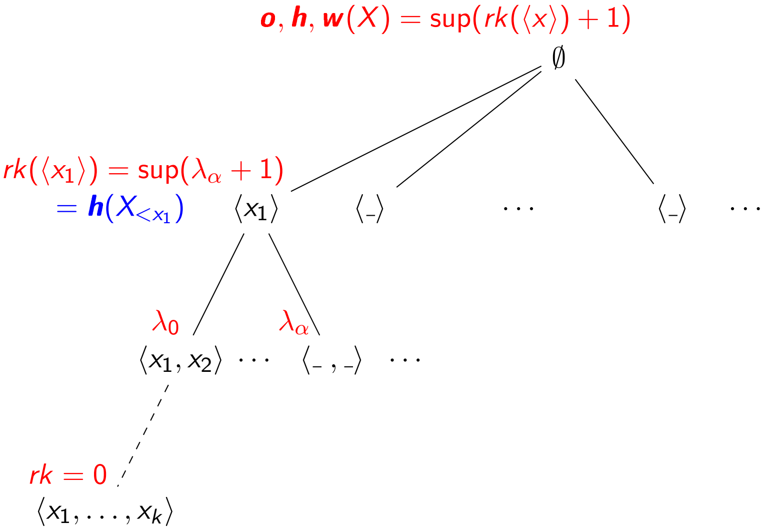


Ordinal invariants

♣ Rank of well-founded trees



Translation into residuals

♦ **Descent equations**

$$\boldsymbol{o}(X) = \sup_{x \in X} \boldsymbol{o}(X_{\not\geq x}) + 1$$

$$\boldsymbol{h}(X) = \sup_{x \in X} \boldsymbol{h}(X_{<x}) + 1$$

$$\boldsymbol{w}(X) = \sup_{x \in X} \boldsymbol{w}(X_{\perp x}) + 1$$

Translation into residuals

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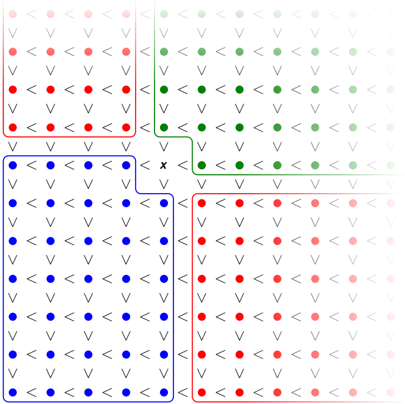

♣ **Ex: Residuals of $\mathbb{N} \times \mathbb{N}$**

In the previous episode. . .

Comparing WQOs

Invariant preserving maps

Let $f : (X, \leq_X) \to (Y, \leq_Y)$ be a map.

♣ Substructures (add points)

Whenever f is injective and $x \leq_X y \Leftrightarrow f(x) \leq_Y f(y)$.

- $X \leq_{\text{st}} Y$ implies $\boldsymbol{h}, \boldsymbol{w}, \boldsymbol{o}(X) \leq \boldsymbol{h}, \boldsymbol{w}, \boldsymbol{o}(Y)$

♣ Augmentations (add relations)

Whenever f is bijective and $f(x) \leq_Y f(y) \Rightarrow x \leq_X y$

- $Y \leq_{\text{aug}} X$ implies $\boldsymbol{w}, \boldsymbol{o}(X) \leq \boldsymbol{w}, \boldsymbol{o}(Y)$

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♦ Condensation (simulates decreasing sequences of Y in X)

Whenever f is surjective, monotone, and
$\forall y \leq_Y f(x), \exists x' \leq_X x$ such that $y = f(x')$

- $X \geq_{cond} Y$ implies $\boldsymbol{h}(X) \geq \boldsymbol{h}(Y)$

In the previous episode. . .

How to compute invariants compositionally

How to compute invariants compositionally

Space	M.O.T.	Height	Width
$A \sqcup B$	$\boldsymbol{o}(A) \oplus \boldsymbol{o}(B)$	$\max(\boldsymbol{h}(A), \boldsymbol{h}(B))$	$\boldsymbol{w}(A) \oplus \boldsymbol{w}(B)$
$A + B$	$\boldsymbol{o}(A) + \boldsymbol{o}(B)$	$\boldsymbol{h}(A) + \boldsymbol{h}(B)$	$\max(\boldsymbol{w}(A), \boldsymbol{w}(B))$
$A \times B$	$\boldsymbol{o}(A) \otimes \boldsymbol{o}(B)$	$\boldsymbol{h}(A) \hat{\oplus} \boldsymbol{h}(B)$	?
$A \cdot B$	$\boldsymbol{o}(A) \cdot \boldsymbol{o}(B)$	$\boldsymbol{h}(A) \cdot \boldsymbol{h}(B)$	$\boldsymbol{w}(A) \odot \boldsymbol{w}(B)$
$\mathrm{M}^\diamond(A)$	$\omega^{\widehat{\boldsymbol{o}(A)}}$	$\boldsymbol{h}^*(A)$	?
A^*	$\omega^{\omega^{(\boldsymbol{o}(X)^{\pm})}}$	$\boldsymbol{h}^*(A)$	$\omega^{\omega^{(\boldsymbol{o}(X)^{\pm})}}$
$\mathcal{P}_{\mathrm{fin}}(A)$	?	?	?

♣ Taken from Džamonja, Schmitz & Schnoebelen(2020)

How to compute invariants compositionally

Space	M.O.T.	Height	Width
$A \sqcup B$	$\boldsymbol{o}(A) \oplus \boldsymbol{o}(B)$	$\max(\boldsymbol{h}(A), \boldsymbol{h}(B))$	$\boldsymbol{w}(A) \oplus \boldsymbol{w}(B)$
$A + B$	$\boldsymbol{o}(A) + \boldsymbol{o}(B)$	$\boldsymbol{h}(A) + \boldsymbol{h}(B)$	$\max(\boldsymbol{w}(A), \boldsymbol{w}(B))$
$A \times B$	$\boldsymbol{o}(A) \otimes \boldsymbol{o}(B)$	$\boldsymbol{h}(A) \hat{\oplus} \boldsymbol{h}(B)$	*Not functional*
$A \cdot B$	$\boldsymbol{o}(A) \cdot \boldsymbol{o}(B)$	$\boldsymbol{h}(A) \cdot \boldsymbol{h}(B)$	$\boldsymbol{w}(A) \odot \boldsymbol{w}(B)$
$\mathrm{M}^\diamond(A)$	$\omega^{\widehat{\boldsymbol{o}(A)}}$	$\boldsymbol{h}^*(A)$	$\omega^{\widehat{\boldsymbol{o}(A)}-1}$
A^*	$\omega^{\omega^{(\boldsymbol{o}(X)^{\pm})}}$	$\boldsymbol{h}^*(A)$	$\omega^{\omega^{(\boldsymbol{o}(X)^{\pm})}}$
$\mathcal{P}_{\mathrm{fin}}(A)$	*Not functional*	*Not functional*	*Not functional*

♣ **Taken from Džamonja, Schmitz & Schnoebelen(2020)**

♦ **Ordinal measures of the set of finite multisets (V. 2023)**

In the previous episode...

Ordinal invariants of $\mathcal{P}_{\text{fin}}$

Finite Powerset

♣ Hoare's embedding

We consider $(\mathcal{P}_{\text{fin}}(X), \leq_{\mathcal{H}})$, with

$$S \leq_{\mathcal{H}} S' \text{ iff } \forall x \in S, \exists y \in S', x \leq y$$

♦ Useful to know

- $\mathcal{P}_{\text{fin}}(\alpha) = 1 + \alpha$ for any ordinal α
- $\mathcal{P}_{\text{fin}}(A \sqcup B) = \mathcal{P}_{\text{fin}}(A) \times \mathcal{P}_{\text{fin}}(B)$

Bounds on width and m.o.t. of $\mathcal{P}_{\text{fin}}$

Theorem

$$1 + \boldsymbol{o}(A) \leq \boldsymbol{o}(\mathcal{P}_{\text{fin}}(A)) \leq 2^{\boldsymbol{o}(A)}$$
$$1 + h(A) \leq \boldsymbol{h}(\mathcal{P}_{\text{fin}}(A)) \leq 2^{\boldsymbol{h}(A)}$$
$$2^{\boldsymbol{w}(A)} \leq \boldsymbol{w}(\mathcal{P}_{\text{fin}}(A)) \leq (\boldsymbol{o}(\mathcal{P}_{\text{fin}}(A))$$

Corollary

$$\boldsymbol{w}(A) = \boldsymbol{o}(A) \Rightarrow \boldsymbol{w}(\mathcal{P}_{\text{fin}}(A)) = \boldsymbol{o}(\mathcal{P}_{\text{fin}}(A)) = 2^{\boldsymbol{o}(A)}$$

♦ **Is the condition $w = o$ met frequently ?**

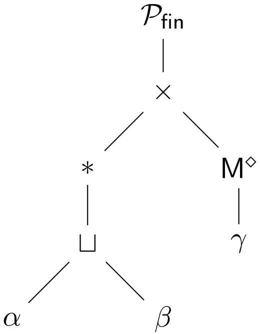
Elementary WQOs: Width and Maximal order type

Is $w = o$ frequent ?

♣ An algebra of elementary wqos (First draft)

$$A, B := \alpha \mid A \sqcup B \mid A + B \mid A \times B \mid A \cdot B \mid \mathrm{M}^{\diamond}(A) \mid A^{*} \mid \mathcal{P}_{\mathsf{fin}}(A)$$

- Basic blocks: linear orderings, i.e., ordinals α
- Closure by usual operations on WQOs



Is $w = o$ frequent ?

$$A, B := \alpha \mid A \sqcup B \mid A + B \mid A \times B \mid A \cdot B \mid \mathrm{M}^\diamond(A) \mid A^* \mid \mathcal{P}_{\mathrm{fin}}(A)$$

Space	M.O.T.	Width	$\boldsymbol{w} = \boldsymbol{o}$?	
α	α	1	✗	
$A \sqcup B$	$\boldsymbol{o}(A) \oplus \boldsymbol{o}(B)$	$\boldsymbol{w}(A) \oplus \boldsymbol{w}(B)$	✓⇒ ✓	
$A + B$	$\boldsymbol{o}(A) + \boldsymbol{o}(B)$	$\max(\boldsymbol{w}(A), \boldsymbol{w}(B))$	✗	
$A \times B$	$\boldsymbol{o}(A) \otimes \boldsymbol{o}(B)$	?	?	
$A \cdot B$	$\boldsymbol{o}(A) \cdot \boldsymbol{o}(B)$	$\boldsymbol{w}(A) \odot \boldsymbol{w}(B)$	✓⇒ ✓	
$\mathrm{M}^\diamond(A)$	$\omega^{\widehat{\boldsymbol{o}(A)}}$	$\omega^{\widehat{\boldsymbol{o}(A)}-1}$	(✓)	if $\boldsymbol{o}(A) \geq \omega$
A^*	$\omega^{\omega^{(\boldsymbol{o}(X)^{\pm})}}$	$\omega^{\omega^{(\boldsymbol{o}(X)^{\pm})}}$	(✓)	if $\boldsymbol{o}(A) \geq 2$
$\mathcal{P}_{\mathrm{fin}}(A)$	$\leq 2^{\boldsymbol{o}(A)}$	$\geq 2^{\boldsymbol{w}(A)}$	✓⇒ ✓	

Is $w = o$ frequent (in elementary WQOs)?

$$A, B := \alpha \geq \omega \mid A \sqcup B \mid A + B \mid A \times B \mid A \cdot B \mid \mathrm{M}^\diamond(A) \mid A^* \mid \mathcal{P}_{\mathrm{fin}}(A)$$

Space	M.O.T.	Width	$\boldsymbol{w} = \boldsymbol{o}$?
$\alpha \geq \omega$	α	1	✗
$A \sqcup B$	$\boldsymbol{o}(A) \oplus \boldsymbol{o}(B)$	$\boldsymbol{w}(A) \oplus \boldsymbol{w}(B)$	✓⇒ ✓
$A + B$	$\boldsymbol{o}(A) + \boldsymbol{o}(B)$	$\max(\boldsymbol{w}(A), \boldsymbol{w}(B))$	✗
$A \times B$	$\boldsymbol{o}(A) \otimes \boldsymbol{o}(B)$	?	?
$A \cdot B$	$\boldsymbol{o}(A) \cdot \boldsymbol{o}(B)$	$\boldsymbol{w}(A) \odot \boldsymbol{w}(B)$	✓⇒ ✓
$\mathrm{M}^\diamond(A)$	$\omega^{\widehat{\boldsymbol{o}(A)}}$	$\omega^{\widehat{\boldsymbol{o}(A)}-1}$	✓
A^*	$\omega^{\omega^{(\boldsymbol{o}(X)^{\pm})}}$	$\omega^{\omega^{(\boldsymbol{o}(X)^{\pm})}}$	✓
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Is $w = o$ frequent ?

$$A, B := \alpha \geq \omega \mid A \sqcup B \mid \cancel{A + B} \mid A \times B \mid \cancel{A \cdot B} \mid \mathrm{M}^\diamond(A) \mid A^* \mid \mathcal{P}_{\text{fin}}(A)$$

Space	M.O.T.	Width	$w = o$?
$\alpha \geq \omega$	α	1	[✓]
$A \sqcup B$	$\boldsymbol{o}(A) \oplus \boldsymbol{o}(B)$	$\boldsymbol{w}(A) \oplus \boldsymbol{w}(B)$	[✓]
$A \times B$	$\boldsymbol{o}(A) \otimes \boldsymbol{o}(B)$	?	?
$\mathrm{M}^\diamond(A)$	$\omega^{\widehat{\boldsymbol{o}(A)}}$	$\omega^{\widehat{\boldsymbol{o}(A)}-1}$	✓
A^*	$\omega^{\omega^{(\boldsymbol{o}(X)^{\pm})}}$	$\omega^{\omega^{(\boldsymbol{o}(X)^{\pm})}}$	✓
$\mathcal{P}_{\text{fin}}(A)$	$\leq 2^{\boldsymbol{o}(A)}$	$\geq 2^{\boldsymbol{w}(A)}$	✓ ⇒ ✓

[✓]: rewriting rule $\begin{cases} \mathcal{P}_{\text{fin}}(\alpha) & \to 1 + \alpha = \alpha \\ \mathcal{P}_{\text{fin}}(A \sqcup B) & \to \mathcal{P}_{\text{fin}}(A) \times \mathcal{P}_{\text{fin}}(B) \end{cases}$

Is $w = o$ frequent ?

$$A, B := \alpha \geq \omega \mid A \sqcup B \mid A \times B \mid \mathrm{M}^{\diamond}(A) \mid A^* \mid \mathcal{P}_{\mathrm{fin}}(A)$$

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$\alpha \geq \omega$	α	1	[✓]
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$A \times B$	$\boldsymbol{o}(A) \otimes \boldsymbol{o}(B)$	?	?
$\mathrm{M}^{\diamond}(A)$	$\omega^{\widehat{\boldsymbol{o}(A)}}$	$\omega^{\widehat{\boldsymbol{o}(A)}-1}$	✓
A^*	$\omega^{\omega^{(\boldsymbol{o}(X)^{\pm})}}$	$\omega^{\omega^{(\boldsymbol{o}(X)^{\pm})}}$	✓
$\mathcal{P}_{\mathrm{fin}}(A)$	$\leq 2^{\boldsymbol{o}(A)}$	$\geq 2^{\boldsymbol{w}(A)}$	✓ $\Rightarrow$ ✓

♦ **What about the cartesian product ?**

Elementary WQOs: Width and Maximal order type

Zooming in on the cartesian product

Width of the Cartesian Product

♣ **A Note on Dilworth's Theorem in the Infinite Case, Abraham(87)**

Let $\alpha_i = \omega^{\beta_i} \cdot k_i + \sigma_i$, with $\sigma_i < \omega^{\beta_i}$

Theorem (Cartesian product of 2 ordinals)

$$\boldsymbol{w}(\alpha_1 \times \alpha_2) = \omega^{1+(\beta_1-1)\oplus(\beta_2-1)} \cdot (k_1 + k_2 - 1) + [\boldsymbol{w}(\omega^{\beta_1} \times \sigma_2) \oplus \boldsymbol{w}(\omega^{\beta_2} \times \sigma_1)]$$

Width of the Cartesian Product

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$$\begin{aligned} \boldsymbol{w}(\alpha_1 \times \alpha_2) = {} & \omega^{1+(\beta_1-1)\oplus(\beta_2-1)} \cdot (k_1 + k_2 - 1) \\ & + [\boldsymbol{w}(\omega^{\beta_1} \times \sigma_2) \oplus \boldsymbol{w}(\omega^{\beta_2} \times \sigma_1)] \end{aligned}$$

Theorem (Cartesian product of n ordinals)

$$\begin{aligned} \boldsymbol{w}(\alpha_1 \times \cdots \times \alpha_n) = {} & \omega^{1+(\beta_1-1)\oplus\cdots\oplus(\beta_n-1)} \cdot \left(\prod k_i - \prod (k_i - 1)\right) \\ & + \bigoplus_{\emptyset \neq I \subsetneq [1,n]} \boldsymbol{w}((\times_{i \notin I} \omega^{\beta_i}) \times (\times_{i \in I} \sigma_i)) \end{aligned}$$

When $w = o$ for the cartesian product

♦ α verifies $\begin{cases} \text{CP1} & \text{if } \alpha = \omega^\beta \text{ with } \beta > 0, \\ \text{CP2} & \text{if } \alpha = \omega^\omega \cdot \gamma \text{ with } \gamma > 0. \end{cases}$

Theorem (Conditions for $w = o$, CP of ordinals)

$$\boldsymbol{w}(\alpha_1 \times \cdots \times \alpha_n) = \boldsymbol{o}(\alpha_1 \times \cdots \times \alpha_n)$$

if there are $i \leq n$ and $j \neq j' \leq n$ such that

- *α_i verifies CP1,*
- *and α_j and $\alpha_{j'}$ verify CP2.*

When $w = o$ for the cartesian product

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Theorem (Conditions for $\boldsymbol{w} = \boldsymbol{o}$, CP of wqos)

$$\boldsymbol{w}(A_1 \times \cdots \times A_n) = \boldsymbol{o}(A_1 \times \cdots \times A_n)$$

if there are $i \leq n$ and $j \neq j' \leq n$ such that

- $\boldsymbol{o}(A_i)$ *verifies CP1,*
- *and* $\boldsymbol{o}(A_j)$ *and* $\boldsymbol{o}(A_{j'})$ *verify CP2.*

When $w = o$ for the cartesian product

♦ α verifies $\begin{cases} \text{CP1} & \text{if } \alpha = \omega^\beta \text{ with } \beta > 0, \\ \text{CP2} & \text{if } \alpha = \omega^\omega \cdot \gamma \text{ with } \gamma > 0. \end{cases}$

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$$\mathbf{w}(A_1 \times \cdots \times A_n) = \mathbf{o}(A_1 \times \cdots \times A_n)$$

if there are $i \leq n$ and $j \neq j' \leq n$ such that

- $\mathbf{o}(A_i)$ *verifies CP1,*
- *and $\mathbf{o}(A_j)$ and $\mathbf{o}(A_{j'})$ verify CP2.*

♦ **On the cartesian product of well-orderings (V. 2022)**

Elementary WQOs: Width and Maximal order type

CP1 and CP2

Is CP1 frequent ?

$$A, B := \alpha \geq \omega \mid A \sqcup B \mid A \times B \mid \mathrm{M}^\diamond(A) \mid A^* \mid \mathcal{P}_{\mathsf{fin}}(A)$$

Space	M.O.T.	CP1	
α	α	(✓)	if $\alpha = \omega^{\alpha'}$
$A \sqcup B$	$\boldsymbol{o}(A) \oplus \boldsymbol{o}(B)$	✗	
$A \times B$	$\boldsymbol{o}(A) \otimes \boldsymbol{o}(B)$	[✓]	
$\mathrm{M}^\diamond(A)$	$\omega^{\widehat{\boldsymbol{o}(A)}}$	✓	
A^*	$\omega^{\omega^{(\boldsymbol{o}(X)^{\pm})}}$	✓	
$\mathcal{P}_{\mathsf{fin}}(A)$	$2^{\boldsymbol{o}(A)}$	(✓)	if $\boldsymbol{w}(A) = \boldsymbol{o}(A)$

♦ **CP1** $\boldsymbol{o} = \omega^\beta$

Is CP1 frequent ?

$$A, B := \alpha = \omega^{\alpha'} \geq \omega \mid A \sqcup B \mid A \times B \mid \mathrm{M}^\diamond(A) \mid A^* \mid \mathcal{P}_{\mathrm{fin}}(A)$$

Space	**M.O.T.**	CP1	
α	α	✓	
$A \sqcup B$	$\boldsymbol{o}(A) \oplus \boldsymbol{o}(B)$	[✓]	
$A \times B$	$\boldsymbol{o}(A) \otimes \boldsymbol{o}(B)$	[✓]	
$\mathrm{M}^\diamond(A)$	$\omega^{\widehat{\boldsymbol{o}(A)}}$	✓	
A^*	$\omega^{\omega^{(\boldsymbol{o}(X)^{\pm})}}$	✓	
$\mathcal{P}_{\mathrm{fin}}(A)$	$2^{\boldsymbol{o}(A)}$	(✓)	if $\boldsymbol{w}(A) = \boldsymbol{o}(A)$

♦ **CP1** $\boldsymbol{o} = \omega^\beta$

[✓] $A \times (B \sqcup C) = (A \times B) \sqcup (A \times C)$

What about CP2 ?

$$A, B := \alpha = \omega^{\alpha'} \geq \omega \mid A \sqcup B \mid A \times B \mid \mathrm{M}^{\diamond}(A) \mid A^* \mid \mathcal{P}_{\mathrm{fin}}(A)$$

Space	M.O.T.	CP1	CP2	
α	α	✓	(✓)	if $\alpha \geq \omega^\omega$
$A \sqcup B$	$\boldsymbol{o}(A) \oplus \boldsymbol{o}(B)$	[✓]	[✓]	
$A \times B$	$\boldsymbol{o}(A) \otimes \boldsymbol{o}(B)$	[✓]	[✓]	
$\mathrm{M}^{\diamond}(A)$	$\omega^{\widehat{\boldsymbol{o}(A)}}$	✓	✓	
A^*	$\omega^{\omega^{(\boldsymbol{o}(X)^{\pm})}}$	✓	✓	
$\mathcal{P}_{\mathrm{fin}}(A)$	$2^{\boldsymbol{o}(A)}$	(✓)	(✓)	if $\boldsymbol{w}(A) = \boldsymbol{o}(A)$

- **CP1** $\boldsymbol{o} = \omega^\beta$
- **CP2** $\boldsymbol{o} = \omega^\omega \cdot \gamma$

What about CP2 ?

$$A, B := \alpha = \omega^{\alpha'} \geq \omega^{\omega} \mid A \sqcup B \mid A \times B \mid \mathrm{M}^{\diamond}(A) \mid A^* \mid \mathcal{P}_{\mathrm{fin}}(A)$$

Space	**M.O.T.**	CP1	CP2	
α	α	✓	✓	
$A \sqcup B$	$\boldsymbol{o}(A) \oplus \boldsymbol{o}(B)$	[✓]	[✓]	
$A \times B$	$\boldsymbol{o}(A) \otimes \boldsymbol{o}(B)$	[✓]	[✓]	
$\mathrm{M}^{\diamond}(A)$	$\omega^{\widehat{\boldsymbol{o}(A)}}$	✓	✓	
A^*	$\omega^{\omega^{(\boldsymbol{o}(X)^{\pm})}}$	✓	✓	
$\mathcal{P}_{\mathrm{fin}}(A)$	$2^{\boldsymbol{o}(A)}$	(✓)	(✓)	if $\boldsymbol{w}(A) = \boldsymbol{o}(A)$

- **CP1** $\boldsymbol{o} = \omega^{\beta}$
- **CP2** $\boldsymbol{o} = \omega^{\omega} \cdot \gamma$

$$A, B := \alpha = \omega^{\alpha'} \geq \omega^{\omega} \mid A \sqcup B \mid A \times B \mid \mathrm{M}^{\diamond}(A) \mid A^* \mid \mathcal{P}_{\mathrm{fin}}(A)$$

Space	**M.O.T.**	$\boldsymbol{w} = \boldsymbol{o}$	CP1	CP2
α	α	[✓]	✓	✓
$A \sqcup B$	$\boldsymbol{o}(A) \oplus \boldsymbol{o}(B)$	[✓]	[✓]	[✓]
$A \times B$	$\boldsymbol{o}(A) \otimes \boldsymbol{o}(B)$	✓	[✓]	[✓]
$\mathrm{M}^{\diamond}(A)$	$\omega^{\widehat{\boldsymbol{o}(A)}}$	✓	✓	✓
A^*	$\omega^{\omega^{(\boldsymbol{o}(X)^{\pm})}}$	✓	✓	✓
$\mathcal{P}_{\mathrm{fin}}(A)$	$2^{\boldsymbol{o}(A)}$	✓	✓	✓

[✓]: $$\begin{cases} \mathcal{P}_{\mathrm{fin}}(\alpha) & \to \alpha \\ \mathcal{P}_{\mathrm{fin}}(A \sqcup B) & \to \mathcal{P}_{\mathrm{fin}}(A) \times \mathcal{P}_{\mathrm{fin}}(B) \\ A \times (B \sqcup C) & \to (A \times B) \sqcup (A \times C) \end{cases}$$

Conclusion (for now)

♣ **We know how to compute $\boldsymbol{w}$ and $\boldsymbol{o}$ of elementary WQOs !**

$$A, B := \alpha = \omega^{\alpha'} \geq \omega^{\omega} \,\big|\, A \sqcup B \,\big|\, A \times B \,\big|\, \mathrm{M}^{\diamond}(A) \,\big|\, A^* \,\big|\, \mathcal{P}_{\mathrm{fin}}(A)$$

- $\boldsymbol{w} = \boldsymbol{o}$ for elementary A...
- ...if A is not linear,
- ...and not a disjoint sum.

Elementary WQOs: Height

Approximated maximal order type

Height of $\mathcal{P}_{\text{fin}}$

Theorem

$$1 + \boldsymbol{h}(X) \leq \boldsymbol{h}(\mathcal{P}_{\text{fin}}(X)) \leq 2^{h(X)}$$

♣ A word about $\mathcal{P}$ and ideals

- $\boldsymbol{h}(\mathcal{P}(X)) = \boldsymbol{o}(X) + 1$
- $\mathcal{P}_{\text{fin}}(Idl(X)) = \mathcal{P}(X)$

♣ Definition of $Idl(X)$

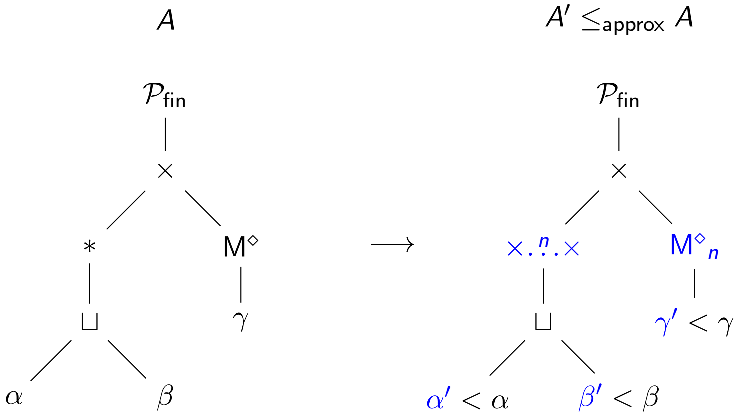
- Ideal: downward-closed, non-empty, directed subset
- $(Idl(X), \subseteq)$

Theorem (Approximated maximal order type)

$$\boldsymbol{h}(\mathcal{P}_{\text{fin}}(A)) \;=\; \underline{\text{o}}(A) \;=\; \sup_{A' \leq_{\text{approx}} A} (\boldsymbol{o}(A') + 1)$$

Space A	$A' \leq_{\text{approx}} A$
α	$\alpha' < \alpha$
$A \sqcup B$	$A' \sqcup B'$
$A \times B$	$A' \times B'$
$\text{M}^{\diamond}(A)$	$\text{M}^{\diamond}{}_n(A')$
A^*	$\overbrace{A' \times \cdots \times A'}^{n}$
$\mathcal{P}_{\text{fin}}(A)$	$\mathcal{P}_{\text{fin}}(A')$

Illustration



Proof Idea

Theorem (Approximated maximal order type)

$$\boldsymbol{h}(\mathcal{P}_{\mathrm{fin}}(A)) \;=\; \underline{\mathrm{o}}(A) \;=\; \sup_{A' \leq_{\mathrm{approx}} A} (\boldsymbol{o}(A') + 1)$$

♣ **A word about $\mathcal{P}$ and ideals**

- $\boldsymbol{h}(\mathcal{P}(X)) = \boldsymbol{o}(X) + 1$
- $\mathcal{P}_{\mathrm{fin}}(Idl(X)) = \mathcal{P}(X)$

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♦ **Proof idea $(\geq)$**

$$\begin{aligned} A' \leq_{\text{approx}} A \;&\Rightarrow\; A' \leq_{\text{st}} A \\ &\Rightarrow Idl(A') \leq_{\text{approx}} A \\ &\Rightarrow Idl(A') \leq_{\text{st}} A \end{aligned}$$

Proof Idea

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- Residual equation: $\boldsymbol{h}(\mathcal{P}_{\text{fin}}(A)) = \sup_{S \in \mathcal{P}_{\text{fin}}(A)}(\boldsymbol{h}(\mathcal{P}_{\text{fin}}(A)_{<S}) + 1)$

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- Residual equation: $\boldsymbol{h}(\mathcal{P}_{\mathrm{fin}}(A)) = \sup_{S \in \mathcal{P}_{\mathrm{fin}}(A)} (\boldsymbol{h}(\mathcal{P}_{\mathrm{fin}}(A)_{<S}) + 1)$
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Proof Idea

Theorem (Approximated maximal order type)

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- Condensation: $A \leq_{cond} B$ implies $\boldsymbol{h}(A) \leq \boldsymbol{h}(B)$

What is $\underline{o}$?

Theorem (Approximated maximal order type)

$$\boldsymbol{h}(\mathcal{P}_{\mathrm{fin}}(A)) \;=\; \underline{o}(A) \;=\; \sup_{A' \leq_{\mathrm{approx}} A} (\boldsymbol{o}(A') + 1)$$

Space A	$A' \leq_{\mathrm{approx}} A$	$\boldsymbol{o}(A')$
α	$\alpha' < \alpha$	α'
$A \sqcup B$	$A' \sqcup B'$	$\boldsymbol{o}(A') \oplus \boldsymbol{o}(B')$
$A \times B$	$A' \times B'$	$\boldsymbol{o}(A') \otimes \boldsymbol{o}(B')$
A^*	$\overbrace{A' \times \cdots \times A'}^{n}$	$\overbrace{\boldsymbol{o}(A') \otimes \cdots \otimes \boldsymbol{o}(A')}^{n}$
$\mathrm{M}^\diamond(A)$	$\mathrm{M}^\diamond{}_n(A')$	like A^*
$\mathcal{P}_{\mathrm{fin}}(A)$	$\mathcal{P}_{\mathrm{fin}}(A')$	$2^{\boldsymbol{w}(A')} \leq \cdots \leq 2^{\boldsymbol{o}(A')}$

Multiplicative indecomposable ordinal

Definition (Multiplicative indecomposable)

$$\alpha \text{ indecomposable } \Leftrightarrow \alpha = \omega^{\omega^{\alpha'}} \text{ for some } \alpha'$$
$$\Leftrightarrow \beta \otimes \gamma < \alpha \text{ for all } \beta, \gamma < \alpha$$

♣ Elementary WQOs (final definition)

$$A, B := \alpha \geq \omega^{\omega} \text{ indecomposable} \mid A \sqcup B \mid A \times B \mid \mathrm{M}^{\diamond}(A) \mid A^* \mid \mathcal{P}_{\text{fin}}(A)$$

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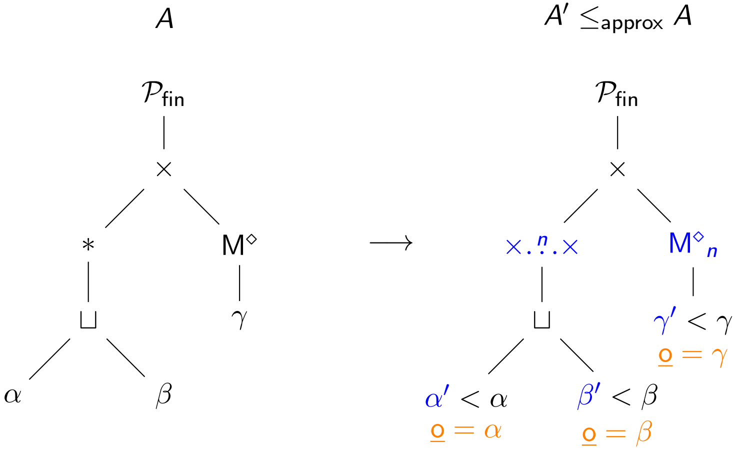
♦ Inductively, $\underline{o}$ stays indecomposable !

Computing o on our example

Assume $\alpha \geq \beta, \gamma$ indecomposable



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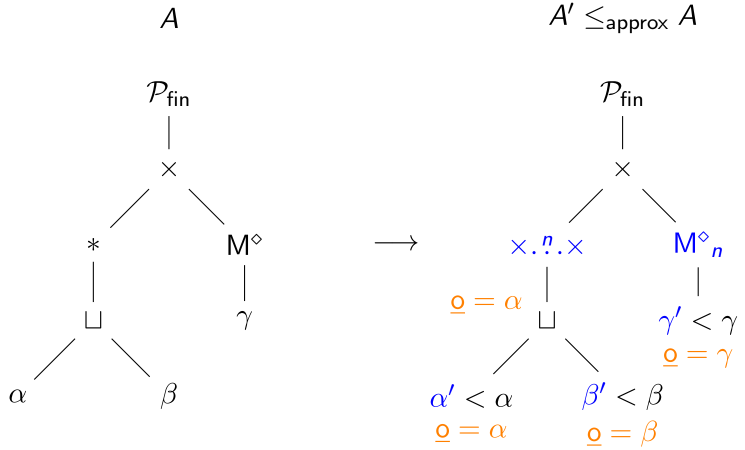

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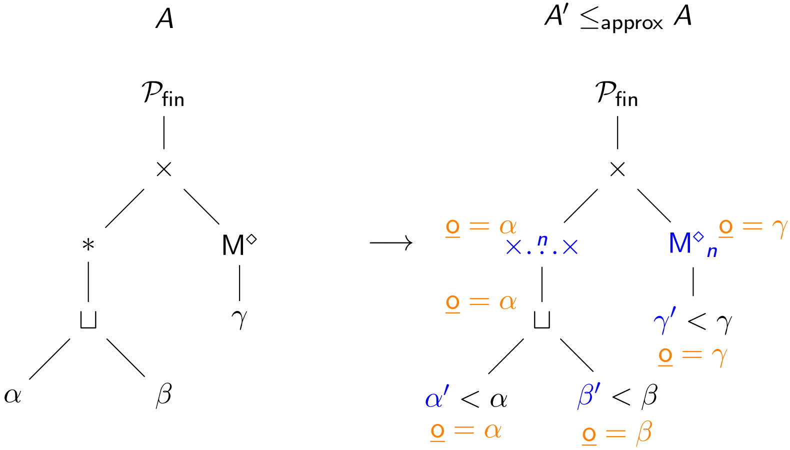

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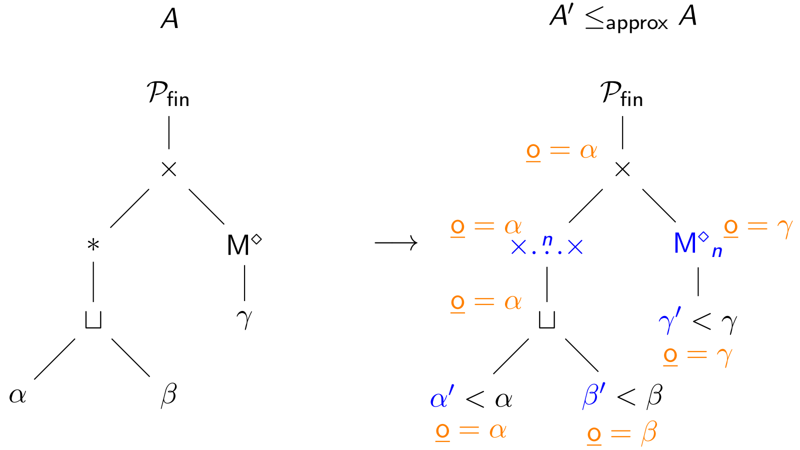

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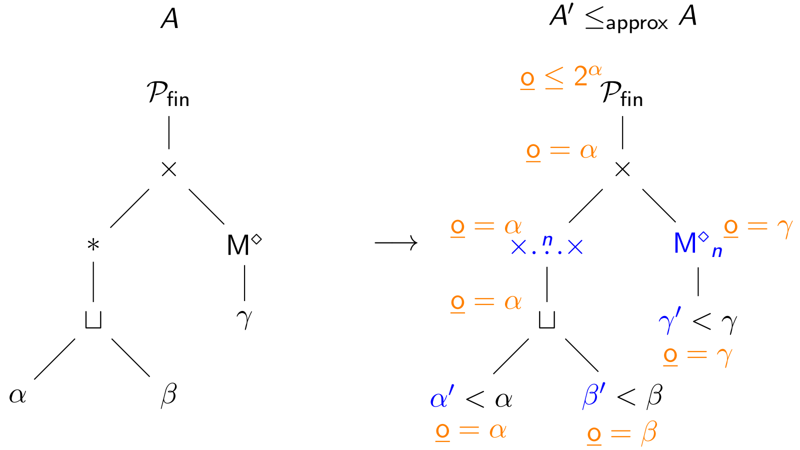

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Theorem (Approximated maximal order type)

$$\boldsymbol{h}(\mathcal{P}_{\text{fin}}(A)) \;=\; \underline{o}(A) \;=\; \sup_{A' \leq_{\text{approx}} A} (\boldsymbol{o}(A') + 1)$$

Space A	$A' \leq_{\text{approx}} A$	$\underline{o}(A)$	
α	$\alpha' < \alpha$	α	
$A \sqcup B$	$A' \sqcup B'$	$\max(\underline{o}(A), \underline{o}(B))$	
$A \times B$	$A' \times B'$	$\max(\underline{o}(A), \underline{o}(B))$	
$\mathrm{M}^{\diamond}(A)$	${\mathrm{M}^{\diamond}}_n(A')$	$\underline{o}(A)$	
A^*	$\overbrace{A' \times \cdots \times A'}^{n}$	$\underline{o}(A)$	
$\mathcal{P}_{\text{fin}}(A)$	$\mathcal{P}_{\text{fin}}(A')$	$2^{\underline{o}(A)}$	if $\underline{w}(A) = \underline{o}(A)$

What is $\underline{o}$?

Theorem (Approximated maximal order type)

$$\boldsymbol{h}(\mathcal{P}_{\text{fin}}(A)) = \underline{o}(A) = \sup_{A' \leq_{\text{approx}} A} (\boldsymbol{o}(A') + 1)$$

Space A	$A' \leq_{\text{approx}} A$	$\underline{o}(A)$	$\underline{w} = \underline{o}$?
α	$\alpha' < \alpha$	α	[✓]
$A \sqcup B$	$A' \sqcup B'$	$\max(\underline{o}(A), \underline{o}(B))$	[✓]
$A \times B$	$A' \times B'$	$\max(\underline{o}(A), \underline{o}(B))$	✓
$\mathrm{M}^{\diamond}(A)$	${\mathrm{M}^{\diamond}}_n(A')$	$\underline{o}(A)$	✓
A^*	$\overbrace{A' \times \cdots \times A'}^{n}$	$\underline{o}(A)$	✓
$\mathcal{P}_{\text{fin}}(A)$	$\mathcal{P}_{\text{fin}}(A')$	$2^{\underline{o}(A)}$	✓ ⇒ ✓

Conclusion

♣ **Elementary WQOs (Final version)**

$$A, B := \alpha \geq \omega^{\omega} \text{ indecomposable} \mid A \sqcup B \mid A \times B \mid \mathrm{M}^{\diamond}(A) \mid A^{*} \mid \mathcal{P}_{\mathrm{fin}}(A)$$

- $\boldsymbol{o}(A)$ indecomposable except if A is a disjoint sum,
- $\boldsymbol{w}(A) = \boldsymbol{o}(A)$ except if A is linear or a disjoint sum,
- $\boldsymbol{h}(\mathcal{P}_{\mathrm{fin}}(A)) =$ max of the ordinals that appear in its expression if A can be expressed without $\mathcal{P}_{\mathrm{fin}}$

Conclusion

Space	M.O.T.	Width	Height
$\alpha \geq \omega$	α	1	α
$A \sqcup B$	$\boldsymbol{o}(A) \oplus \boldsymbol{o}(B)$	$\boldsymbol{w}(A) \oplus \boldsymbol{w}(B)$	$\max(\boldsymbol{h}(A), \boldsymbol{h}(B))$
$A \times B$	$\boldsymbol{o}(A) \otimes \boldsymbol{o}(B)$	$\boldsymbol{o}(A) \otimes \boldsymbol{o}(B)$	$\boldsymbol{h}(A) \hat{\oplus} \boldsymbol{h}(B)$
$\mathrm{M}^{\diamond}(A)$	$\omega^{\widehat{\boldsymbol{o}(A)}}$	$\omega^{\widehat{\boldsymbol{o}(A)}-1}$	$h^*(A)$
A^*	$\omega^{\omega^{(\boldsymbol{o}(X)^{\pm})}}$	$\omega^{\omega^{(\boldsymbol{o}(X)^{\pm})}}$	$h^*(A)$
$\mathcal{P}_{\mathrm{fin}}(A)$	$2^{\boldsymbol{o}(A)}$	$2^{\boldsymbol{o}(A)}$	$\underline{\mathrm{o}}(A)$

♦ **Don't forget to rewrite your expression**

$\mathcal{P}_{\mathrm{fin}}(\alpha), \mathcal{P}_{\mathrm{fin}}(A \sqcup B), A \times (B \sqcup C)$

Conclusion

♦ **Soon on arXiv (hopefully)**

- Bounds on the ordinal invariants $\mathcal{P}_{\mathsf{fin}}$
- Proof of tightness
- A larger elementary family (with ω !)

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On the cartesian product of linear orderings

- Width of the cartesian product of n ordinals
- Handy tool for the width: Quasi-incomparable subsets

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On the cartesian product of linear orderings

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Ordinal measures of the set of finite multisets

- Width of the multiset (embedding order)
- Width and height of the multiset (multiset order)